Pseudo-almost periodic viscosity solutions of second-order nonlinear parabolic equations

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In this paper we generalize the comparison result of Bostan and Namah (2007) [8] to the second-order parabolic case and prove two properties of pseudo-almost periodic functions; then by using Perron’s method we prove the existence and uniqueness of time pseudo-almost periodic viscosity solutions of second-order parabolic equations under usual hypotheses.

1. Introduction

In this paper we will study the time pseudo-almost periodic viscosity solutions of second-order parabolic equations of the form

\[
\begin{align*}
\partial_t u + H(x, u, Du, D^2u) &= f(t), \quad (x, t) \in \Omega \times \mathbb{R}, \\
u(x, t) &= 0, \quad (x, t) \in \partial \Omega \times \mathbb{R}
\end{align*}
\]

where \( \Omega \in \mathbb{R}^N \) is a bounded open subset and \( \partial \Omega \) is its boundary. Here \( H \) and \( f \) are continuous functions, and \( f \) is pseudo-almost periodic in \( t \). Pseudo-almost periodic functions are a new generalization of almost periodic functions. They were introduced by Zhang [1]. In [1], Zhang also discussed their applications to some differential equations. After that, some of the literature discussed pseudo-almost periodic solutions for various differential equations, for example [2–7]. Bostan and Namah have studied the time periodic and almost periodic viscosity solutions of first-order Hamilton–Jacobi equations in the paper [8]; Nunziante considered the existence and uniqueness of viscosity solutions of parabolic equations with discontinuous time dependence in [9,10], but the time pseudo-almost periodic viscosity solutions of second-order parabolic equations have not been studied as far as we know. We will first prove two properties of pseudo-almost periodic functions, then use Perron’s method to prove the existence of time pseudo-almost periodic viscosity solutions of (1). Perron’s method was introduced by Ishii [11] in the proof of the existence of viscosity solutions of first-order Hamilton–Jacobi equations; Crandall et al. used applications of Perron’s method to second-order differential equations in the paper [12], but these did not apply in the parabolic case.
To study the existence and uniqueness of viscosity solutions of (1), we will use some results on the Cauchy–Dirichlet problem of the form

\[
\begin{aligned}
&\partial_t u + H(x, t, u, Du, D^2u) = 0, \quad \text{in } \Omega \times (0, T), \\
&u(x, t) = 0, \quad \text{for } x \in \partial \Omega \text{ and } 0 \leq t < T, \\
&u(x, 0) = u_0(x), \quad \text{for } x \in \Omega.
\end{aligned}
\]  

(2)

where \(u_0(x) \in C(\overline{\Omega})\) is given. Crandall et al. studied the comparison result for the Cauchy–Dirichlet problem in [12], and it follows the maximum principle of Crandall and Ishii [13].

This paper is structured as follows. In Section 2, we present the definition and some properties of almost periodic functions and pseudo-almost periodic functions. In Section 3, first we list some hypotheses and some results that will be used for establishing the existence and uniqueness of viscosity solutions; here we give an improvement of a comparison result from paper [8], to fit second-order parabolic equations, and then we prove the uniqueness and existence of time pseudo-almost periodic viscosity solutions. At the end, we concentrate on the asymptotic behavior of time pseudo-almost periodic solutions for large frequencies.

2. Pseudo-almost periodic function

In this section we first recall the definition and some properties of almost periodic functions. For more details on the theory of almost periodic functions one can refer to Corduneanu [14].

Proposition 2.1. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function. The following conditions are equivalent:

1. \( \forall \varepsilon > 0, \exists \varepsilon(\varepsilon) > 0 \) such that \( \forall a \in \mathbb{R}, \exists r \in [a, a + l(\varepsilon)) \) satisfying
   \[
   |f(t + \tau) - f(t)| < \varepsilon, \quad \forall t \in \mathbb{R};
   \]  
   \[
   (3)
   
   2. \( \forall \varepsilon > 0, \) there is a trigonometric polynomial \( T_\varepsilon(t) = \sum_{k=1}^{n} \{a_k \cdot \cos(\lambda_k t) + b_k \cdot \sin(\lambda_k t)\} \) where \( a_k, b_k, \lambda_k \in \mathbb{R}, 1 \leq k \leq n, \) such that \( |f(t) - T_\varepsilon(t)| < \varepsilon, \forall t \in \mathbb{R}; \)

3. for all real sequences \((h_k)_n\) there is a subsequence \((h_{n_k})_k\) such that \( (f(\cdot + h_{n_k}))_k \) converges uniformly on \( \mathbb{R}. \)

Definition 2.2. We say that a continuous function \( f \) is almost periodic iff \( f \) satisfies one of the three conditions of Proposition 2.1.

A number \( \tau \) verifying (3) is called \( \varepsilon \) almost periodic. By using Proposition 2.1 we get the following property of almost periodic functions.

Proposition 2.3. Assume that \( f : \mathbb{R} \to \mathbb{R} \) is almost periodic. Then \( f \) is a bounded uniformly continuous function.

Proposition 2.4. Assume that \( f : \mathbb{R} \to \mathbb{R} \) is almost periodic. Then \( \frac{1}{T} \int_{a}^{a+T} f(t) \, dt \) converges as \( T \to +\infty \) uniformly with respect to \( a \in \mathbb{R}. \) Moreover the limit does not depend on \( a \) and it is called the average of \( f: \)

\[
\exists(f) := \lim_{T \to +\infty} \frac{1}{T} \int_{a}^{a+T} f(t) \, dt, \quad \text{uniformly w.r.t. } a \in \mathbb{R}.
\]

Proposition 2.5. Assume that \( f : \mathbb{R} \to \mathbb{R} \) is almost periodic and denote by \( F \) a primitive of \( f. \) Then \( F \) is almost periodic iff \( F \) is bounded.

For the goal of applications to the differential equations, Yoshizawa [15] extended almost periodic functions to so-called uniformly almost periodic functions.

Definition 2.6 ([15]). We say that \( u : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) is almost periodic in \( t \) uniformly with respect to \( x \) if \( u \) is continuous in \( t \) uniformly with respect to \( x \) and \( \forall \varepsilon > 0, \exists \varepsilon(\varepsilon) > 0 \) such that all intervals of length \( l(\varepsilon) \) contain a number \( \tau \) which is \( \varepsilon \) almost periodic for \( u(x, \cdot), \forall x \in \overline{\Omega}:

\[
|u(x, t + \tau) - u(x, t)| < \varepsilon, \quad \forall (x, t) \in \overline{\Omega} \times \mathbb{R}.
\]

In the paper [1], the author defines a new type of almost periodic function, i.e. a pseudo-almost periodic function.

Let \( X \) be a Banach space. Let \( \Omega \) be a closed subset of \( X \) and let \( \varphi(\mathbb{R}) \) (respectively, \( \varphi(\Omega \times \mathbb{R}) \)) be the space of bounded continuous complex-valued functions on \( \mathbb{R} \) (respectively, \( \Omega \times \mathbb{R} \)) with a supremum norm. Suppose that \( J \in \{\mathbb{R}^+, \mathbb{R}\} \) and let \( C(\Omega \times J, X) \) (respectively, \( C(J, X) \)) be the space of bounded, continuous functions from \( J \times \Omega \) (respectively, \( J \)) to \( X \) with supremum norm.
Set
\[ \text{PAP}_0(\mathbb{R}) = \left\{ \varphi \in \varphi(\mathbb{R}) : \lim_{t \to \infty} \frac{1}{2t} \int_{-t}^{t} |\varphi(s)|ds = 0 \right\} \]
and
\[ \text{PAP}_0(\Omega \times \mathbb{R}) = \left\{ \varphi \in \varphi(\Omega \times \mathbb{R}) : \lim_{t \to \infty} \frac{1}{2t} \int_{-t}^{t} |\varphi(Z, s)|ds = 0 \text{ uniformly in } Z \in \Omega \right\}. \]

**Definition 2.7.** A function \( f \in \varphi(\mathbb{R}) (\varphi(\Omega \times \mathbb{R})) \) is called pseudo-almost periodic if
\[ f = g + \varphi, \]
where \( g \in \text{AP}(\mathbb{R}) (\text{AP}(\Omega \times \mathbb{R})) \) and \( \varphi \in \text{PAP}_0(\mathbb{R}) (\text{PAP}_0(\Omega \times \mathbb{R})) \). The functions \( g \) and \( \varphi \) are called the almost periodic component and the ergodic perturbation respectively of the function \( f \). Denote by \( \text{PAP}(\mathbb{R}) (\text{PAP}(\Omega \times \mathbb{R})) \) the set of all such functions \( f \).

**Definition 2.8.** A continuous function \( f(t) \in L(\mathbb{R}) \) is called ergodic if there exists a constant \( M(f) \) such that
\[ \lim_{t \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t + s)dt = M(f) \quad \text{uniformly for } s \in \mathbb{R}. \]

**Lemma 2.9.** Suppose that \( f(t) \in L(\mathbb{R}) \); the following conditions are equivalent:
1. \( f(t) \) is ergodic;
2. \[ \lim_{u \to \infty} \frac{1}{u} \int_{0}^{u} f(t + s)dt = M(f) \quad \text{uniformly for } s \in \mathbb{R}; \]
3. \[ \lim_{u \to \infty} \frac{1}{u} \int_{-u}^{0} f(t + s)dt = M(f) \quad \text{uniformly for } s \in \mathbb{R}; \]
4. \[ \lim_{u \to \infty} \frac{1}{u} \int_{0}^{u} f(s - t)dt = M(f) \quad \text{uniformly for } s \in \mathbb{R}. \]

**Definition 2.10.** A closed subset \( C \) of \( J \) is said to be an ergodic zero set in \( R \) if \( m(C \cap [a, t])/(2t) \to 0 \) as \( t \to \infty \) where \( a = 0 \) when \( J = \mathbb{R}^+ \) and \( a = -t \) when \( J = \mathbb{R} \), and \( m \) is the Lebesgue measure on \( \mathbb{R} \).

As for the numerical case, \( \varphi \in \text{PAP}(J, X) \) is in \( \text{PAP}_0(J, X) \) if and only if for \( \varepsilon > 0 \), the set \( C_\varepsilon = \{ t \in J : \|\varphi(t)\| \geq \varepsilon \} \) is an ergodic zero set in \( J \).

**Theorem 2.11.** A function \( f \in C(J, X) \) is pseudo-almost periodic if and only if for each \( \varepsilon > 0 \) there are a number \( \delta > 0 \), a relatively dense subset \( P_\varepsilon \) and an ergodic zero subset \( C_\varepsilon \) of \( J \) such that
\[ \|f(t) - f(t + \tau)\| < \varepsilon(t \in P_\varepsilon, t, t + \tau \in J \setminus C_\varepsilon) \]
and
\[ \|f(t') - f(t'')\| < \varepsilon(t', t'' \in J \setminus C_\varepsilon, |t' - t''| < \delta). \]

Suppose that \( h(t) = \int_{-\infty}^{t} e^\gamma(\sigma - t)f(\sigma)d\sigma \), where \( \gamma > 0 \) is a constant, and \( t \in \mathbb{R} \). Then we have:

**Proposition 2.12.** If \( f(t) \in \text{PAP}(\mathbb{R}) \), then \( h(t) \in \text{PAP}(\mathbb{R}) \).

**Proof.** As \( f(t) \in \text{PAP}(\mathbb{R}) \), then
\[ f = g + \varphi, \]
where $g \in AP(\mathbb{R})$ and $\varphi \in PAP_0(\mathbb{R})$. Next we will prove that $h(t) \in PAP(\mathbb{R})$.

$$h(t) = \int_{-\infty}^{t} e^{\varphi(a-t)} f(\sigma) d\sigma$$
$$= \int_{-\infty}^{t} e^{\varphi(a-t)} g(\sigma) d\sigma + \int_{-\infty}^{t} e^{\varphi(a-t)} \varphi(\sigma) d\sigma \quad \text{(4)}$$

Suppose that $I(t) = \int_{-\infty}^{t} e^{\varphi(a-t)} g(\sigma) d\sigma$, $II(t) = \int_{-\infty}^{t} e^{\varphi(a-t)} \varphi(\sigma) d\sigma$. By the proposition of almost periodic functions and $g \in AP(\mathbb{R})$, we know that $I(t) \in AP(\mathbb{R})$. Then we just need to prove that $II(t) \in PAP_0(\mathbb{R})$ to deduce that $h(t) \in PAP(\mathbb{R})$. In fact,

$$\frac{1}{2T} \int_{-T}^{T} \| II(t) \| dt \leq \frac{1}{2T} \int_{-T}^{T} dt \int_{-\infty}^{t} e^{\varphi(a-t)} |\varphi(\sigma)| d\sigma$$
$$= \frac{1}{2T} \int_{-\infty}^{t} |\varphi(\sigma)| d\sigma \int_{-T}^{T} e^{\varphi(a-t)} dt + \frac{1}{2T} \int_{-T}^{T} |\varphi(\sigma)| d\sigma \int_{-T}^{T} e^{\varphi(a-t)} dt$$
$$= II_1 + II_2. \quad \text{(5)}$$

Set $\| \varphi \| = \sup_{t \in \mathbb{R}} |\varphi(t)|$; then

$$II_1 = \frac{1}{2T} \int_{-\infty}^{t} |\varphi(\sigma)| d\sigma \int_{-T}^{T} e^{\varphi(a-t)} dt \leq \frac{1}{2T} \frac{1}{\gamma^2} \| \varphi \| (1 - e^{-2\gamma T}).$$

So $II_1 \to 0$ when $T \to \infty$.

$$II_2 = \frac{1}{2T} \int_{-\infty}^{t} |\varphi(\sigma)| d\sigma \int_{-T}^{T} e^{\varphi(a-t)} dt = \frac{1}{2T} \frac{1}{\gamma} \int_{-T}^{T} |\varphi(\sigma)| (1 - e^{\varphi(a-T)}) d\sigma.$$

Since $s < T$, $1 - e^{\varphi(a-T)}$ is bounded, and $\varphi \in PAP_0(\mathbb{R})$, so $II_2 \to 0$ when $T \to \infty$. Then we prove that $h(t) \in PAP(\mathbb{R})$. \qed

**Proposition 2.13.** Assume that $f(t)$ is pseudo-almost periodic. Then $\frac{1}{T} \int_{a}^{a+T} f(t) dt$ converges as $T \to +\infty$ uniformly with respect to $a \in R$. Moreover the limit does not depend on $a$ and it is called the average of $f$:

$$\Im(f) := \lim_{T \to +\infty} \frac{1}{T} \int_{a}^{a+T} f(t) dt, \quad \text{uniformly with respect to } a \in R.$$  

**Proof.** As $f(t) \in PAP(\mathbb{R})$, then

$$f = g + \varphi,$$

where $g \in AP(\mathbb{R})$ and $\varphi \in PAP_0(\mathbb{R})$. Since

$$\lim_{T \to +\infty} \frac{1}{T} \int_{a}^{a+T} f(t) dt = \lim_{T \to +\infty} \frac{1}{T} \int_{a}^{a+T} g(t) dt + \lim_{T \to +\infty} \frac{1}{T} \int_{a}^{a+T} \varphi(t) dt,$$

and also we have $-|\varphi(t)| \leq \varphi(t) \leq |\varphi(t)|$, so

$$-\lim_{T \to +\infty} \frac{1}{T} \int_{a}^{a+T} |\varphi(t)| dt \leq \lim_{T \to +\infty} \frac{1}{T} \int_{a}^{a+T} \varphi(t) dt \leq \lim_{T \to +\infty} \frac{1}{T} \int_{a}^{a+T} |\varphi(t)| dt,$$

and then by definition of $\varphi(t)$ and Lemma 2.9 we have

$$\lim_{T \to +\infty} \frac{1}{T} \int_{a}^{a+T} \varphi(t) dt = 0.$$

From Proposition 2.4 we know that

$$\langle g \rangle := \lim_{T \to +\infty} \frac{1}{T} \int_{a}^{a+T} g(t) dt, \quad \text{uniformly with respect to } a \in R.$$

Thus there exists $\langle f \rangle$ such that

$$\lim_{T \to +\infty} \frac{1}{T} \int_{a}^{a+T} f(t) dt = \langle f \rangle = \langle g \rangle, \quad \text{uniformly with respect to } a \in R. \quad \square$$
**Definition 2.14.** We say that $f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is pseudo-almost periodic in $t$ uniformly with respect to $x$ if

$$f = g + \varphi$$

where $g \in \text{AP}(\mathbb{R}^N \times \mathbb{R})$ and $\varphi \in \text{PAP}_0(\mathbb{R}^N \times \mathbb{R})$. Denote by PAP(\mathbb{R}^N \times \mathbb{R}) the set of all such functions.

### 3. Pseudo-almost periodic viscosity solutions

In this section we get some results for pseudo-almost periodic viscosity solutions.

We consider the following two equations to get some results used for the existence and uniqueness of pseudo-almost periodic viscosity solutions, that is, the Dirichlet problems of the form

$$
\begin{align*}
\partial_t u + H(x, t, u, Du, D^2 u) &= 0, & \text{in } \Omega \times (0, T), \\
u(x, t) &= 0, & \text{for } x \in \partial \Omega \text{ and } 0 \leq t < T
\end{align*}
$$

and

$$
\begin{align*}
F(x, u, Du, D^2 u) &= 0, & \text{in } \Omega, \\
u &= 0, & \text{on } \partial \Omega,
\end{align*}
$$

where in Eq. (7), $\Omega$ is an arbitrary open subset of $\mathbb{R}^N$.

In [12], Crandall et al. proved such a theorem.

**Theorem 3.1 ([12]).** Let $\Theta_i$ be a locally compact subset of $\mathbb{R}^N_i$ for $i = 1, \ldots, k$:

$$\Theta = \Theta_1 \times \cdots \times \Theta_k,$$

suppose that $u_i \in \text{USC}(\Theta_i)$, and let $\varphi$ be twice continuously differentiable in a neighborhood of $\Theta$. Set

$$w(x) = u_1(x_1) + \cdots + u_k(x_k) \text{ for } x = (x_1, \ldots, x_k) \in \Theta,$$

and suppose that $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_k) \in \Theta$ is a local maximum of $w - \varphi$ relative to $\Theta$. Then for each $\epsilon > 0$ there exists $X_i \in S(N_i)$ such that

$$(D_{x_i} \varphi(\hat{x}), X_i) \in f_{\Theta_i}^{\epsilon} u_i(\hat{x}_i) \text{ for } i = 1, \ldots, k,$$

and the block diagonal matrix with entries $X_i$ satisfies

$$
- \left( \frac{1}{\epsilon} + \|A\| \right) I \leq \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{pmatrix} \leq A + A^2
$$

where $A = D^2 \varphi(\hat{x}) \in S(N), N = N_1 + \cdots + N_k$.

Put $k = 2$, $\Theta_1 = \Theta_2 = \Omega$, $u_1 = u$, $u_2 = -v$, $\varphi(x, y) = (\alpha/2)|x - y|^2$, where $\alpha > 0$, and recall that $f_{\Omega}^{2-} v = -f_{\Omega}^{2+}(-v)$; then, from Theorem 3.1, at a local maximum $(\hat{x}, \hat{y})$ of $u(x) - v(y) - \varphi(x, y)$, we have

$$D_x \varphi(\hat{x}, \hat{y}) = -D_y \varphi(\hat{x}, \hat{y}) = \alpha(\hat{x} - \hat{y}), \quad A = \alpha \begin{pmatrix} 1 & -I \\ -I & 1 \end{pmatrix}, \quad A^2 = 2\alpha A, \quad \|A\| = 2\alpha.$$

We conclude that for each $\epsilon > 0$, there exists $X, Y \in S(N)$ such that

$$(\alpha(\hat{x} - \hat{y}), X) \in f_{\Omega}^{2+} u(\hat{x}), \quad (\alpha(\hat{x} - \hat{y}), Y) \in f_{\Omega}^{2-} v(\hat{y})$$

and

$$- \left( \frac{1}{\epsilon} + 2\alpha \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \alpha(1 + 2\epsilon\alpha) \begin{pmatrix} 1 & -I \\ -I & 1 \end{pmatrix}.$$

Choosing $\epsilon = 1/\alpha$ one can get

$$- 3\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} 1 & -I \\ -I & 1 \end{pmatrix}.$$

To prove the existence and uniqueness of viscosity solutions, let us see the following main hypotheses first.

As in [12], we present a fundamental monotonicity condition of $H$, i.e.

$$H(x, r, p, X) \leq H(x, s, p, Y) \text{ whenever } r \leq s \text{ and } Y \leq X,$$
where $r, s \in \mathbb{R}, x \in \Omega, p \in \mathbb{R}^N, X, Y \in S(N)$ and $S(N)$ is a set of symmetric $N \times N$ matrices. Then we will say that $H$ is proper.

Assume that there exists $\gamma > 0$ such that
\begin{equation}
\gamma (r - s) \leq H(x, r, p, X) - H(x, s, p, X), \quad \text{for } r \geq s, (x, p, X) \in \overline{\Omega} \times \mathbb{R}^N \times S(N),
\end{equation}
and there is a function $\omega : [0, \infty] \to [0, \infty]$ that satisfies $\omega(0+) = 0$ such that
\begin{equation}
\begin{cases}
H(y, r, \alpha(x - y), Y) - H(x, r, \alpha(x - y), X) \leq \omega(\alpha|x - y|^2 + |x - y|)
\end{cases}
\end{equation}
whenever $x, y \in \Omega, r \in \mathbb{R}, X, Y \in S(N)$, and (9) holds.

Now we can easily prove the following result. There is a similar result for first-order Hamilton–Jacobi equations in the book of Barles [16].

Lemma 3.2. Assume that $H \in C(\overline{\Omega} \times (0, T] \times \mathbb{R} \times \mathbb{R}^N \times S(N))$ and $u \in C(\overline{\Omega} \times (0, T])$ is a viscosity subsolution (resp. supersolution) of $\partial_t u + H(x, t, u, Du, D^2 u) = 0$, $(x, t) \in \Omega \times (0, T)$. Then $u$ is a viscosity subsolution (resp. supersolution) of $\partial_t u + H(x, t, u, Du, D^2 u) = 0$, $(x, t) \in \Omega \times (0, T)$.

Proof. Since $u \in C(\overline{\Omega} \times (0, T])$ is a viscosity subsolution of $\partial_t u + H(x, t, u, Du, D^2 u) = 0$, $(x, t) \in \Omega \times (0, T)$, $\forall \psi \in C^2(\Omega \times (0, T))$ and for the local maximum $(\hat{x}, \hat{t}) \in (\Omega \times (0, T))$ for $u - \psi$, we have
\begin{equation}
\partial_t \psi(\hat{x}, \hat{t}) + H(\hat{x}, \hat{t}, u(\hat{x}, \hat{t}), D\psi(\hat{x}, \hat{t}), D^2 \psi(\hat{x}, \hat{t})) \leq 0.
\end{equation}
Now we prove that if $(x_0, T)$ is a local maximum of $u - \psi$ in $\Omega \times (0, T)$, then
\begin{equation}
\partial_t \psi(x_0, T) + H(x_0, T, u(x_0, T), D\psi(x_0, T), D^2 \psi(x_0, T)) \leq 0.
\end{equation}
Suppose that $(x_0, T)$ is a strict local maximum of $u - \psi$ in $\Omega \times (0, T)$; we consider the function
\begin{equation}
\psi_\varepsilon(x, t) = u(x, t) - \psi(x, t) - \varepsilon (T - t)^{-1},
\end{equation}
for small $\varepsilon > 0$. Then we know that the function $\psi_\varepsilon(x, t)$ has a local maximum point $(x_\varepsilon, t_\varepsilon)$ such that $t_\varepsilon < T$ and $(x_\varepsilon, t_\varepsilon) \to (x_0, T)$ when $\varepsilon \to 0$. So we deduce that at the point $(x_\varepsilon, t_\varepsilon)$,
\begin{equation}
\partial_t \psi_\varepsilon(x_\varepsilon, t_\varepsilon) + \frac{\varepsilon}{(T - t_\varepsilon)^2} + H(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), D\psi(x_\varepsilon, t_\varepsilon), D^2 \psi(x_\varepsilon, t_\varepsilon)) \leq 0.
\end{equation}
As the term $\frac{\varepsilon}{(T - t_\varepsilon)^2}$ is positive, we obtain
\begin{equation}
\partial_t \psi(x_\varepsilon, t_\varepsilon) + H(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), D\psi(x_\varepsilon, t_\varepsilon), D^2 \psi(x_\varepsilon, t_\varepsilon)) \leq 0.
\end{equation}
The results following upon supposing that $\varepsilon \to 0$. This process can be easily applied to the viscosity supersolution case. \hfill \Box

By time periodicity one gets:

Proposition 3.3. Assume that $H \in C(\overline{\Omega} \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \times S(N))$ and $u \in C(\overline{\Omega} \times \mathbb{R})$ are $T$ periodic such that $u$ is a viscosity subsolution (resp. supersolution) of $\partial_t u + H(x, t, u, Du, D^2u) = 0$, $(x, t) \in \Omega \times (0, T)$. Then $u$ is a viscosity subsolution (resp. supersolution) of $\partial_t u + H(x, t, u, Du, D^2u) = 0$, $(x, t) \in \Omega \times (0, T)$.

Crandall et al. have proved the following two comparison results.

Theorem 3.4 ([12]). Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$, and $F \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times S(N))$ be proper and satisfy (11), (12). Let $u \in \text{USC}(\overline{\Omega})$ (respectively, $v \in \text{LSC}(\overline{\Omega})$) be a subsolution (respectively, supersolution) of $F = 0$ in $\Omega$ and $u \leq v$ on $\partial \Omega$. Then $u \leq v$ in $\Omega$.

Theorem 3.5 ([12]). Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Let $H \in C(\overline{\Omega} \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \times S(N))$ be continuous, proper and satisfy (12) for each fixed $t \in [0, T)$, with the same function $\omega$. If $u$ is a subsolution of (2) and $v$ is a supersolution of (2), then $u \leq v$ on $[0, T) \times \Omega$.

We generalize the comparison result in article [8] for first-order Hamilton–Jacobi equations, and get two theorems for second-order parabolic equations. Let us see a proposition that we will need in the proof of the comparison result (see [12]).

Proposition 3.6 ([12]). Let $\Theta$ be a subset of $\mathbb{R}^N$, $\Phi \in \text{USC}(\Theta)$, $\Psi \in \text{LSC}(\Theta)$, $\Psi \geq 0$ and $M_\alpha = \sup_{\Theta} (\Phi(x) - \alpha \Psi(x))$
\begin{equation}
M_\alpha \neq \infty \Rightarrow \lim_{\alpha \to \infty} M_\alpha < \infty \quad \text{and} \quad x_\alpha \in \Theta \quad \text{be chosen such that}
\end{equation}
\begin{equation}
\lim_{\alpha \to \infty} (M_\alpha - (\Phi(x_\alpha) - \alpha \Psi(x_\alpha))) = 0.
\end{equation}
Then the following holds:

\[
\begin{align*}
(\text{i}) \quad \lim_{\alpha \to \infty} \alpha \psi(x_0) &= 0, \\
(\text{ii}) \quad \Psi(x) = 0 \quad \text{and} \quad \lim_{\alpha \to \infty} M_{\alpha} &= \Phi(x) = \sup_{\Theta} \Phi(x) \quad \text{whenever} \ \hat{x} \in \Theta \text{is a limit point of} \ x_0 \text{as} \ \alpha \to \infty. \quad (13)
\end{align*}
\]

**Remarks 3.7.** In Proposition 3.6, when \( M, \Theta, x, \Phi(x), \Psi(x) \) are replaced by \( 2N, \Theta \times \Theta, (x,y), u(x) - v(y), (1/2)|x - y|^2 \) respectively, we can get the following results:

\[
\begin{align*}
(\text{i}) \quad \lim_{\alpha \to \infty} \alpha |x_\alpha - y_\alpha|^2 &= 0, \\
(\text{ii}) \quad \Psi(x) &= 0 \quad \text{and} \quad \lim_{\alpha \to \infty} M_{\alpha} = u(\hat{x}) - v(\hat{x}) = \sup_{\Theta} (u(x) - v(x)) \quad \text{whenever} \ \hat{x} \in \Theta \text{is a limit point of} \ x_0 \text{as} \ \alpha \to \infty. \quad (14)
\end{align*}
\]

Now we have:

**Theorem 3.8.** Let \( \Omega \in \mathbb{R}^N \) be open and bounded. Assume that \( H \in C(\overline{\Omega} \times [0,T]) \times \mathbb{R}^N \times S(N) \) is continuous, proper and satisfies \((11),(12)\) for each fixed \( t \in [0,T] \). Let \( u, v \) be a bounded u.s.c. subsolution of \( \partial_t u + H(x,t,u,Du,D^2u) = f(x,t) \) in \( \Omega \times (0,T) \), \( u(x,0) = 0 \) for \( x \in \partial \Omega \) and \( 0 \leq t < T \) or a l.s.c. supersolution of \( \partial_t v + H(x,t,v,Dv,D^2v) = g(x,t) \) in \( \Omega \times (0,T) \), \( v(x,0) = 0 \) for \( x \in \partial \Omega \) and \( 0 \leq t < T \), where \( f, g \in \text{BUC}(\overline{\Omega} \times [0,T]) \).

\[
\lim_{t \to 0} (u(x,t) - u(x,0))_+ = \lim_{t \to 0} (v(x,t) - v(x,0))_- = 0, \quad \text{uniformly for} \ x \in \overline{\Omega},
\]

and

\[
u(\cdot,0) \in \text{BUC}(\overline{\Omega}) \quad \text{or} \quad u(\cdot,0) \in \text{BUC}(\overline{\Omega}).
\]

Then we have for all \( t \in [0,T] \),

\[
e^{\gamma t} \|(u(\cdot,t) - v(\cdot,t))\|_{L^\infty(\overline{\Omega})} \leq \|u(\cdot,0) - v(\cdot,0)\|_{L^\infty(\overline{\Omega})} + \int_0^t e^{\gamma s} \|(f(\cdot,s) - g(\cdot,s))\|_{L^\infty(\overline{\Omega})} ds,
\]

where \( \gamma = \gamma_0, R_0 = \max\{\|u\|_{L^\infty(\overline{\Omega} \times [0,T])}, \|v\|_{L^\infty(\overline{\Omega} \times [0,T])}\} \).

**Proof.** Let us consider the function given by

\[
w_\alpha(x,y,t) = u(x,t) - v(y,t) - \psi(x,y,t),
\]

where \( \psi(x,y,t) = \frac{\gamma}{2}(|x-y|^2 + \phi(t)) \), and \( \phi(t) \in C^1([0,T]) \). As we know that \( u \) and \( v \) are bounded semicontinuous in \( \overline{\Omega} \times [0,T] \) and \( \Omega \in \mathbb{R}^N \) is open and bounded, we can find \( (\hat{x}(t_\alpha),\hat{y}(t_\alpha)) \in \overline{\Omega} \times \overline{\Omega} \) for \( t_\alpha \in [0,T] \) such that \( M_{\alpha}(t_\alpha) := \sup_{\overline{\Omega} \times \overline{\Omega}} (u(x,t_\alpha) - v(y,t_\alpha)) \) \( \text{and} \quad \psi(x,y,t_\alpha) \quad \text{for} \ x \in \partial \Omega \forall t \in (0,T) \). Here without loss of generality, we can assume that \( M_{\alpha}(t_\alpha) = 0 \). Since \( \overline{\Omega} \times \overline{\Omega} \times [0,T] \) is compact, these maxima \( (\hat{x}(t_\alpha),\hat{y}(t_\alpha),t_\alpha) \) converge to a point of the form \( (z(t),z(t),t) \) from Remark 3.7. From Theorem 3.1 and the discussion following it, there exists \( X_\alpha, Y_\alpha \in S(N) \) such that

\[
(\alpha(\hat{x}(t_\alpha) - \hat{y}(t_\alpha)), X_\alpha) \in \int_{\overline{\Omega}}^2 u(\hat{x}(t_\alpha), t_\alpha), \quad (\alpha(\hat{x}(t_\alpha) - \hat{y}(t_\alpha)), Y_\alpha) \in \int_{\overline{\Omega}}^2 - v(\hat{y}(t_\alpha), t_\alpha)
\]

and

\[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \leq \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \leq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]

which implies that \( X_\alpha \leq Y_\alpha \). At the maximum point, from the definition of \( u \) being a subsolution and \( v \) being a supersolution, we arrive at the following:

\[
\partial_t \psi(\hat{x}(t_\alpha), \hat{y}(t_\alpha), t_\alpha) + H(\hat{x}(t_\alpha), t_\alpha, u(\hat{x}(t_\alpha), t_\alpha), \alpha(\hat{x}(t_\alpha) - \hat{y}(t_\alpha)), X_\alpha)
\]

\[= H(\hat{y}(t_\alpha), t_\alpha, v(\hat{y}(t_\alpha), t_\alpha), \alpha(\hat{y}(t_\alpha) - \hat{x}(t_\alpha)), Y_\alpha) \leq f(\hat{x}(t_\alpha), t_\alpha) - g(\hat{y}(t_\alpha), t_\alpha), \]

and by the proper condition of \( H \), we have \( H(\hat{y}(t_\alpha), t_\alpha, v(\hat{y}(t_\alpha), t_\alpha), \alpha(\hat{y}(t_\alpha) - \hat{x}(t_\alpha)), Y_\alpha) \leq H(\hat{y}(t_\alpha), t_\alpha, v(\hat{y}(t_\alpha), t_\alpha), \alpha(\hat{x}(t_\alpha) - \hat{y}(t_\alpha)), X_\alpha)
\]
and hence we get
\[ \partial_t \varphi(\tilde{x}(t_\alpha), \tilde{y}(t_\alpha), t_\alpha) + H(\tilde{y}(t_\alpha), t_\alpha, u(\tilde{x}(t_\alpha), t_\alpha), \alpha(\tilde{x}(t_\alpha) - \tilde{y}(t_\alpha)), X_\alpha) 
- H(\tilde{y}(t_\alpha), t_\alpha, v(\tilde{y}(t_\alpha), t_\alpha), \alpha(\tilde{x}(t_\alpha) - \tilde{y}(t_\alpha)), X_\alpha) - \omega(\alpha|\tilde{x}(t_\alpha) - \tilde{y}(t_\alpha)|^2 + |\tilde{x}(t_\alpha) - \tilde{y}(t_\alpha)|) \leq h(t_\alpha) \]
where \( h(t_\alpha) = f(\tilde{x}(t_\alpha), t_\alpha) - g(\tilde{y}(t_\alpha), t_\alpha), \forall t_\alpha \in [0, T]. \) For any \( t_\alpha \in [0, T] \) consider
\[ r(t_\alpha) = \frac{1}{u(\tilde{x}(t_\alpha), t_\alpha) - v(\tilde{y}(t_\alpha), t_\alpha)} \begin{align*} 
& (H(\tilde{y}(t_\alpha), t_\alpha, u(\tilde{x}(t_\alpha), t_\alpha), \alpha(\tilde{x}(t_\alpha) - \tilde{y}(t_\alpha)), X_\alpha) 
& - \gamma u(\tilde{x}(t_\alpha), t_\alpha) - H(\tilde{y}(t_\alpha), t_\alpha, v(\tilde{y}(t_\alpha), t_\alpha), \alpha(\tilde{x}(t_\alpha) - \tilde{y}(t_\alpha)), X_\alpha) + \gamma v(\tilde{y}(t_\alpha), t_\alpha)).
\end{align*} \]
if \( u(\tilde{x}(t_\alpha), t_\alpha) \neq v(\tilde{y}(t_\alpha), t_\alpha) \), and \( r(t_\alpha) = 0 \) otherwise. From hypothesis Eq. (11) we deduce that \( H(x, t, z, p, X) - \gamma \cdot z \) is nondecreasing with respect to \( z \); then we have \( r(t_\alpha) \geq 0 \) for all \( t_\alpha \in [0, T] \). Hence we have
\[ \begin{align*} 
H(\tilde{y}(t_\alpha), t_\alpha, u(\tilde{x}(t_\alpha), t_\alpha), \alpha(\tilde{x}(t_\alpha) - \tilde{y}(t_\alpha)), X_\alpha) - H(\tilde{y}(t_\alpha), t_\alpha, v(\tilde{y}(t_\alpha), t_\alpha), \alpha(\tilde{x}(t_\alpha) - \tilde{y}(t_\alpha)), X_\alpha) 
& = (\gamma + r(t_\alpha))(u(\tilde{x}(t_\alpha), t_\alpha) - v(\tilde{y}(t_\alpha), t_\alpha)), \forall t_\alpha \in [0, T].
\end{align*} \]
Notice that \( u(\tilde{x}(t_\alpha), t_\alpha) - v(\tilde{y}(t_\alpha), t_\alpha) = \varphi(\tilde{x}(t_\alpha), \tilde{y}(t_\alpha), t_\alpha) \); we get
\[ \partial_t \varphi(\tilde{x}(t_\alpha), \tilde{y}(t_\alpha), t_\alpha) + (\gamma + r(t_\alpha))\varphi(\tilde{x}(t_\alpha), \tilde{y}(t_\alpha), t_\alpha) - \omega(\alpha|\tilde{x}(t_\alpha) - \tilde{y}(t_\alpha)|^2 + |\tilde{x}(t_\alpha) - \tilde{y}(t_\alpha)|) \leq h(t_\alpha). \]
Replacing \( u(\tilde{x}(t_\alpha), t_\alpha) - v(\tilde{y}(t_\alpha), t_\alpha) \) by \( \varphi(\tilde{x}(t_\alpha), \tilde{y}(t_\alpha), t_\alpha) \) in the expression for \( r(t_\alpha) \) we know that \( r(\cdot) \) is integrable and denote by \( A(t_\alpha) \) the function \( A(t_\alpha) = \int_0^{t_\alpha} (\gamma + r(\sigma))d\sigma, t_\alpha \in [0, T]. \) After integration one gets
\[ \varphi(t_\alpha) \leq e^{-A(t_\alpha)}(\varphi(0) + \int_0^{t_\alpha} e^{A(s)} \cdot (h(s) + \omega(\alpha|\tilde{x}(s) - \tilde{y}(s)|^2 + |\tilde{x}(s) - \tilde{y}(s)|))ds), \quad t_\alpha \in [0, T]. \]
Now taking \( u(\tilde{x}(t_\alpha), t_\alpha) - v(\tilde{y}(t_\alpha), t_\alpha) \) instead of \( \varphi(\tilde{x}(t_\alpha), \tilde{y}(t_\alpha), t_\alpha) \) for any \( t_\alpha \in [0, T] \) and supposing that \( \alpha \to \infty \), we can get
\[ u(z(t), t) - v(z(t), t) \leq e^{-A(t)} \left( u(z(0), 0) - v(z(0), 0) + \int_0^t e^{A(s)} \cdot h(s)ds \right), \quad t \in [0, T]. \] (16)
Finally we deduce that for all \( t \in [0, T], \)
\[ e^{\gamma t} \| u(\cdot, t) - v(\cdot, t) \|_{L^\infty(\tilde{T})} \leq \| (u(\cdot), 0) - (v(\cdot), 0) \|_{L^\infty(\tilde{T})} + \int_0^t e^{\gamma s} \| (f(\cdot, s) - g(\cdot, s)) \|_{L^\infty(\tilde{T})} ds. \] (17)
\[ \Box \]

**Theorem 3.9.** Let \( \Omega \in \mathbb{R}^N \) be open and bounded. Assume that \( H : C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times S(\mathbb{N})) \) is continuous, proper, \( T \) periodic and satisfies (11), (12). Let \( u \) be a bounded time periodic viscosity u.s.c. subsolution of \( \partial_t u + H(x, t, u, Du, D^2u) = f(x, t) \) in \( \Omega \times \mathbb{R}, u(x, t) = 0 \) for \( (x, t) \in \partial \Omega \times \mathbb{R} \) and \( v \) a bounded time periodic viscosity l.s.c. supersolution of \( \partial_t v + H(x, t, v, Dv, D^2v) = g(x, t) \) in \( \Omega \times \mathbb{R}, v(x, t) = 0 \) for \( (x, t) \in \partial \Omega \times \mathbb{R} \), where \( f, g \in \text{BUC}(\overline{\Omega} \times \mathbb{R}) \). Then we have
\[ \sup_{x \in \tilde{T}} (u(x, t) - v(x, t)) \leq \sup_{s \leq t} \int_s^t \sup_{x \in \tilde{T}} (f(x, \sigma) - g(x, \sigma)) d\sigma. \]

**Proof.** As the proof of **Theorem 3.8**, we get Eq. (16)
\[ u(z(t), t) - v(z(t), t) \leq e^{-A(t)} \left( u(z(0), 0) - v(z(0), 0) + \int_0^t e^{A(s)} \cdot h(s)ds \right), \quad t \in [0, T]. \]
We introduce that \( F(s) = - \int_s^t h(\sigma)d\sigma, s, t \in [0, T]. \) By integration by parts we have
\[ \int_0^t e^{A(s)}h(s)ds = - \int_0^t e^{A(s)}F(s)ds 
= - \int_0^t h(\sigma)d\sigma + \int_0^t e^{A(s)}A'(s) \int_s^t h(\sigma)d\sigma ds 
\leq - \int_0^t h(\sigma)d\sigma + (e^{A(t)} - 1) \sup_{0 \leq s \leq t} \int_s^t h(\sigma)d\sigma. \]
We deduce that for all $t \in [0, T]$ we have

$$\sup_{x \in \overline{\Omega}} (u(x, t) - v(x, t)) \leq e^{-\gamma t} \sup_{x \in \overline{\Omega}} (u(x, 0) - v(x, 0)) + \sup_{0 \leq s \leq t} \int_0^s \sup_{x \in \overline{\Omega}} (f(x, \sigma) - g(x, \sigma)) d\sigma.$$ 

Like in the proof of Corollary 2.2 in paper [8], we can reach the conclusion. □

In order to prove the existence of a viscosity solution, we recall Perron’s method as follows (see [12,11]). To discuss the method, we assume the following: if $u : \Omega \rightarrow [-\infty, \infty]$, then

$$u^*(x) = \lim_{r \downarrow 0} \sup_{y \in \partial \Omega \cap B_r(x)} \{u(y) : |x - y| \leq r\},$$

$$u_*(x) = \lim_{r \downarrow 0} \inf_{y \in \partial \Omega \cap B_r(x)} \{u(y) : |x - y| \leq r\}.$$ 

\textbf{Theorem 3.10 (Perron’s Method).} Let the comparison hold for (7); i.e., if $w$ is a subsolution of (7) and $v$ is a supersolution of (7), then $w \leq v$. Suppose also that there is a subsolution $u$ and a supersolution $\overline{u}$ of (7) that satisfy the boundary condition $u_*(x) = \overline{u}^*(x) = 0$ for $x \in \partial \Omega$. Then

$$W(x) = \sup \{u(x) : u \leq w \leq \overline{u} \text{ and } w \text{ is a subsolution of (7)}\}$$

is a solution of (7).

From paper [12], we have the following remarks as a supplement to \textbf{Theorem 3.10}.

\textbf{Remarks 3.11 ([12]).} Notice that the subset $\Omega$ in Eq. (7) in some parts of the proof in \textbf{Theorem 3.10} was just open in $\mathbb{R}^N$. In order to generalize this and formulate the version of \textbf{Theorem 3.10} that we will need later, we now make some remarks. Suppose that $\Omega$ is locally compact, and $G^+$, $G^-$ are defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N \times S(N)$ and have the following properties: $G^+$ is upper semicontinuous, $G^-$ is lower semicontinuous, and classical solutions (solutions that are twice continuously differentiable in the pointwise sense) of $G^+ \leq 0$ on a relatively open subset of $\Omega$ are solutions of $G^- \leq 0$. Suppose, moreover, that whenever $u$ is a solution of $G^- \leq 0$ on $\Omega$ and $v$ is a solution of $G^+ \geq 0$ on $\Omega$ we have $u \leq v$ on $\Omega$. Then we conclude that the existence of such a subsolution and supersolution guarantees that there is a unique function $u$, obtained by Perron’s construction, that is a solution of both $G^+ \geq 0$ and $G^- \leq 0$ on $\Omega$.

Now we will prove the uniqueness and existence of pseudo-almost periodic viscosity solutions. For the uniqueness we have the following result.

\textbf{Theorem 3.12.} Let $\Omega \times \mathbb{R}^N$ be open and bounded. Assume that $H \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times S(N))$ is continuous, proper and satisfies (11), (12) for $t \in \mathbb{R}$. Let $u$ a bounded u.s.c. viscosity subsolution of $\partial_t u + H(x, t, u, Du, D^2 u) = f(x, t)$ in $\Omega \times \mathbb{R}$, $u(x, t) = 0$ for $(x, t) \in \partial \Omega \times \mathbb{R}$, and $v$ a bounded l.s.c. viscosity supersolution of $\partial_t v + H(x, t, v, Dv, D^2 v) = g(x, t)$ in $\Omega \times \mathbb{R}$, $v(x, t) = 0$ for $(x, t) \in \partial \Omega \times \mathbb{R}$ where $f, g \in$ BUC($\overline{\Omega} \times \mathbb{R}$). Then we have for all $t \in \mathbb{R}$,

$$\sup_{x \in \overline{\Omega}} (u(x, t) - v(x, t))_+ \leq e^{-\gamma t} \sup_{x \in \overline{\Omega}} (f(x, \sigma) - g(x, \sigma))_+ d\sigma.$$

\textbf{Proof.} Take $t_0$, $t \in \mathbb{R}$, $t_0 \leq t$ and by using \textbf{Theorem 3.8} write for all $x \in \overline{\Omega}$

$$u(x, t) - v(x, t) \leq e^{-\gamma (t-t_0)} \cdot (\|u\|_\infty + \|v\|_\infty) + e^{-\gamma t} \int_{t_0}^t e^{\gamma \sigma} \sup_{x \in \overline{\Omega}} (f(y, \sigma) - g(y, \sigma))_+ d\sigma$$

where $\gamma = \gamma_{t_0}$, $R_0 = \max(\|u\|_\infty, \|v\|_\infty)$. Then the conclusion follows by taking the limit $t_0 \to -\infty$. □

Now we concentrate on the existence part.

\textbf{Theorem 3.13.} Let $\Omega$ be a bounded open subset in $\mathbb{R}^N$. Assume that $H \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times S(N))$ is continuous, proper and satisfies (11), (12). Assume that $f : \mathbb{R} \to \mathbb{R}$ is pseudo-almost periodic and $H(x, -M, 0, 0) \leq f(t) \leq H(x, M, 0, 0)$, $\forall (x, t) \in \Omega \times \mathbb{R}$, where $M > 0$ is a constant. Then there is a time pseudo-almost periodic viscosity solution in BC($\overline{\Omega} \times \mathbb{R}$) of (1).

\textbf{Proof.} Here we consider the problem

\begin{equation}
\begin{cases}
\partial_t u_n + H(x, u_n, Du_n, D^2 u_n) = f(t), & (x, t) \in \Omega \times (-n, +\infty), \\
u_n(x, t) = 0, & (x, t) \in \partial \Omega \times [-n, +\infty), \\
u_n(x, -n) = 0, & x \in \overline{\Omega}
\end{cases}
\end{equation}
for all $n \geq 1$. As we know that $H(x, -M, 0, 0) \leq f(t) \leq H(x, M, 0, 0)$, $\forall (x, t) \in \Omega \times \mathbb{R}$, there exists a viscosity solution $u_n(x, t)$ of (19) from Theorem 3.5 and Remark 3.11. Then we will prove that for all $t \in \mathbb{R}$, $(u_n(t))_{n \geq 1}$ converges to a pseudo-almost periodic viscosity solution of (1). As we already know that $H(x, -M, 0, 0) \leq f(t) \leq H(x, M, 0, 0)$, $\forall (x, t) \in \Omega \times \mathbb{R}$, we can deduce by Theorem 3.5 that $-M \leq u_n(x, t) \leq M$, $\forall (x, t) \in \overline{\Omega} \times [-n, +\infty)$. As $f$ is pseudo-almost periodic, then

\[ f = g + \varphi, \]

where $g \in AP(\mathbb{R})$ and $\varphi \in PAP_0(\mathbb{R})$. For any $\varepsilon > 0$, take $\tau \in P_{\varepsilon_1}$, where $\varepsilon_1 = \frac{\varepsilon}{2}$. Set $C_{\varepsilon_2} = \{ t \in \mathbb{R} : \| \varphi \| \geq \varepsilon_2 \}$, where $\varepsilon_2 = \frac{\varepsilon}{2}$. Like in the proof of Proposition 6.6 in paper [8], we obtain in $\overline{\Omega} \times \mathbb{R} \setminus C_{\varepsilon_2}$ by fixing $t, t + \tau \in \mathbb{R} \setminus C_{\varepsilon_2}$ and $n$ large enough,

\[ |u_n(x, t) - u_n(x, t + \tau)| \leq 2M \cdot e^{-\gamma(t-\tau)} + e^{-\gamma t} \int_{t}^{t+\tau} e^{\gamma \sigma} |f(\sigma + \tau) - f(\sigma)| d\sigma \leq 2M \cdot e^{-\gamma(t-\tau)} + \varepsilon. \]

By taking the limit $n \rightarrow +\infty$ we have $t_n \rightarrow -\infty$ and therefore

\[ |u(x, t) - u(x, t + \tau)| \leq \varepsilon, \quad (x, t) \in \overline{\Omega} \times \mathbb{R} \setminus C_{\varepsilon_2}. \]

For any $\varepsilon > 0$, there exists $\delta > 0$, for every $t \in \mathbb{R} \setminus C_{\varepsilon_2}$, for which we can find $t' \in \mathbb{R} \setminus C_{\varepsilon_2}$ such that $|t' - t| = |t_0| < \delta$. Observe that the function $u_n : \overline{\Omega} \times [-n - t_0, +\infty[ \rightarrow \mathbb{R}$, $u_n(x, t) = u_n(x, t + t_0)$ instead of the function $v_n(x, t)$ as before; then by the uniform continuity of $g$ we conclude in $\overline{\Omega} \times \mathbb{R} \setminus C_{\varepsilon_2}$ that

\[ |u_n(x, t) - u_n(x, t + t_0)| \leq 2M \cdot e^{-\gamma(t-\tau)} + e^{-\gamma t} \int_{t}^{t+t_0} e^{\gamma \sigma} |f(\sigma + t_0) - f(\sigma)| d\sigma \leq 2M \cdot e^{-\gamma(t-\tau)} + \varepsilon. \]

By taking the limit $n \rightarrow +\infty$ we have $t_n \rightarrow -\infty$ and therefore

\[ |u(x, t) - u(x, t')| \leq \varepsilon, \quad (x, t, t') \in \overline{\Omega} \times \mathbb{R} \setminus C_{\varepsilon_2} \times \mathbb{R} \setminus C_{\varepsilon_2}, \quad |t - t'| < \delta. \]

From Theorem 2.11 we conclude that $u(x, t) \in BC(\overline{\Omega} \times \mathbb{R})$ is pseudo-almost periodic. \(\Box\)

Now we discuss the asymptotic behavior of time pseudo-almost periodic viscosity solutions for large frequencies, and there is a similar description for Hamilton–Jacobi equations in paper [8]. Let us observe the following equation:

\[
\begin{aligned}
\partial_t u_n + H(x, u_n, Du_n, D^2 u_n) &= f_n(t), \quad (x, t) \in \Omega \times \mathbb{R}, \\
u_n(x, t) &= 0, \quad (x, t) \in \partial \Omega \times \mathbb{R},
\end{aligned}
\]

(20)

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a pseudo-almost periodic function. For all $n \geq 1$ notice that $f_n(t) = f(nt)$, $\forall t \in \mathbb{R}$ is pseudo-almost periodic and has the same average as $f$. We introduce also the stationary equation

\[
\begin{aligned}
H(x, u, Du, D^2 u) &= (f), \quad x \in \Omega, \\
u(x) &= 0, \quad x \in \partial \Omega.
\end{aligned}
\]

(21)

Now suppose that such a hypothesis exists:

\[ \exists M > 0 \suchthat H(x, -M, 0, 0) \leq f(t), \forall (x, t) \in \Omega \times \mathbb{R}. \]

(22)

**Theorem 3.14.** Let $\Omega \in \mathbb{R}^N$ be open and bounded. Assume that $H \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times S(\mathbb{N}))$ is continuous, proper and satisfies (12) for $t \in \mathbb{R}$ and (22) where $f$ is a pseudo-almost periodic function. Suppose also that there is a bounded l.s.c viscosity supersolution $\tilde{V} \geq -M$ of (21), and that $t \rightarrow F(t) = \int_{0}^{t} (f(s) - f) ds$ is bounded, and denote by $V$ ($v_n$) the minimal stationary (resp. time almost periodic) l.s.c. viscosity supersolution of (21) (resp. (20)). Then the sequence $(v_n)_n$ converges uniformly on $\overline{\Omega} \times \mathbb{R}$ towards $V$ and $\| v_n - V \|_{L^\infty(\overline{\Omega} \times \mathbb{R})} \leq \frac{\varepsilon}{\gamma} \| f \|_{L^\infty(\mathbb{R})}$, $\forall n \geq 1$.

**Proof.** As $v_n = \sup_{\alpha \geq 0} v_n, a$ is pseudo-almost periodic, we introduce $w_{n,a}(x, t) = v_n, a(x, \frac{1}{n} t)$, $(x, t) \in \overline{\Omega} \times \mathbb{R}$, which is also pseudo-almost periodic. As $v_n, a$ satisfies in the viscosity sense $\alpha(v_n, a + M) + \partial_t v_n, a + H(x, w, a, Dv_n, a, D^2 v_n, a) = f_n(t)$, $(x, t) \in \Omega \times \mathbb{R}$, we deduce that $w_{n,a}$ satisfies in the viscosity sense

\[ \alpha(u_n, a + M) + n \partial_t w_{n,a} + H(x, w_n, a, Dw, a, D^2 w_n, a) = f(t), \quad (x, t) \in \Omega \times \mathbb{R}, \]

which can be rewritten as

\[ \partial_t w_{n,a} + \frac{1}{n}(\alpha w_{n,a} + H(x, w_n, a, Dw, a, D^2 w_n, a)) = \frac{1}{n}(f(t) - \alpha M), \quad (x, t) \in \Omega \times \mathbb{R}. \]

Recall also that we have in the viscosity sense

\[ \frac{1}{n}(\alpha V_a + H(x, V_a, D V_a, D^2 V_a)) = \frac{1}{n}(f) - \alpha M, \quad x \in \Omega. \]
By using Theorem 3.9 we deduce that
\[ w_{n,a}(x, t) - V_a(x) \leq \frac{1}{n} \int_s^t (f(\sigma) - \langle f \rangle) \, d\sigma \leq \frac{1}{n} \| F \|_{L^\infty(R)}, \]
and similarly \( V_a(x) - w_{n,a}(x, t) \leq \frac{2}{n} \| F \|_{L^\infty(R)}, \) \( \forall n \geq 1. \) We have for all \( n \leq 1, \)
\[ \left| v_{n,a} \left( x, \frac{t}{n} \right) - V_a(x) \right| \leq \frac{2}{n} \| F \|_{L^\infty(R)}, \]
and after taking the limit \( \alpha \downarrow 0 \) one gets for all \( (x, t) \in \mathcal{M} \times R, \)
\[ \left| v_{n,a} \left( x, \frac{t}{n} \right) - V(x) \right| \leq \frac{2}{n} \| F \|_{L^\infty(R)}. \]
Finally we deduce that \( \| v_n - V \|_{L^\infty(\mathcal{M} \times R)} \leq \frac{2}{n} \| F \|_{L^\infty(R)} \) for all \( n \geq 1. \) \( \square \)

References