The Galerkin method for the KdV equation using a new basis of smooth piecewise cubic polynomials

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Abstract

In this paper, we introduce the Galerkin method using a new basis of smooth piecewise cubic polynomials for the one-dimensional nonlinear KdV equation. The stability analysis shows that the present method is unconditionally stable in $L^2$-norm. Numerical experiments show that the proposed method preserves the conservation laws and is effective for simulating the motions and the interactions of solitary waves.

Article info

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The Galerkin method
Smooth piecewise cubic polynomial
Conservation law
Stability

1. Introduction

Consider the Korteweg–de Vries (KdV) equation

$$u_t + uu_x + l u_{xxx} = 0, \quad x \in I = [a, b], \quad 0 < t \leq T,$$

with the initial condition

$$u(x, 0) = u_0(x), \quad x \in I,$$

and homogeneous boundary conditions

$$u(a, t) = u(b, t) = 0, \quad u_x(a, t) = u_x(b, t) = 0, \quad 0 < t \leq T,$$

where $\varepsilon$ and $\mu$ are positive constants, and the subscripts $x$ and $t$ denote differentiation with respect to $x$ and $t$, respectively, $u_0$ is a prescribed function.

The KdV equation arises in the study of a number of physical problems such as water waves, plasma physics, and anharmonic lattices. The equation was first introduced in 1895 by Korteweg and de Vries [1] and the analytical solutions to particular problems have been given in [2,3]. However, it is possible to find the analytical solutions only for the equation with limited initial and boundary conditions, and, hence, the numerical solutions are very useful to study the physical phenomena. Taha and Ablowitz [4] have introduced some numerical schemes for the KdV equation and there have been numerous investigations to numerically solve the KdV equation, based on the finite difference [5–7], the finite element [8,9], the B-spline Galerkin [10–13], the spectral [14–17], the discontinuous Galerkin [18,19], and the heat balance integral methods [20], etc. Since the KdV equation is third-order, a $C^1$-finite element space should be used for the standard Galerkin method. Thus, the B-spline Galerkin and the spectral methods can be applied to numerically solve the KdV equation, and these methods are very effective (cf. [10–17]). However, the B-spline bases are not nodal bases and the bases for the spectral methods...
do not have local supports, which demand more cost in computation. In this paper, we introduce a new nodal basis of smooth piecewise cubic polynomials for the numerical solution of the KdV equation, and several examples are considered to compare our method with other techniques in [10,12,15,22]. The finite dimensional space spanned by this basis is a subspace of \( H^2(I) \).

A special property of the KdV equation is that its solutions may exhibit solitary waves, known as solitons, which preserve their original size, shape, and velocity after interaction. It has been pointed out that numerical methods that preserve some of the conserved quantities, i.e., invariants, of the given differential equations may show a more accurate behavior in time than those which do not preserve (cf. [21]). We will investigate conservation properties in our numerical experiment to illustrate efficiency of the proposed method.

In Section 2, we introduce the Galerkin scheme using a new basis of smooth piecewise cubic polynomials. The stability of the present method is considered in Section 3, and the results of numerical experiment are shown in Section 4.

2. Numerical method

The variational form of problem Eqs. ((1)-(3)) can be characterized as follows:

Find \( u \in H^1_0(I) \) such that

\[
(u_t, w) - \frac{\varepsilon}{2} (u^2, w_x) - \mu (u_{xx}, w_x) = 0 \quad \forall \; w \in H^1_0(I),
\]

\[
u(x, 0) = u_0(x),
\]

where \((\cdot, \cdot)\) denotes the usual inner product on \( L_2(I) \).

Suppose the interval \( I = [a, b] \) is divided into \( J \) elements \( I_i = [x_{i-1}, x_i] \) (\( i = 1, \ldots, J \)), where \( a = x_0 < x_1 < \cdots < x_J = b \), and let \( h_i = x_i - x_{i-1} \).

Let

\[
\theta_i = \frac{h_{i-1}}{h_i(h_i + h_{i-1})}, \quad \eta_i = \frac{h_i}{h_{i+1}(h_i + h_{i+1})}, \quad i = 1, \ldots, J - 1.
\]

Then the basis \( \{ \phi_0, \phi_1, \ldots, \phi_J \} \) of smooth cubic interpolation polynomials can be defined as follows:

\[
\phi_0(x) = L_0(x) - \frac{1}{h_1} H_0(x) - \theta_1 H_1(x),
\]

\[
\phi_1(x) = L_1(x) + \frac{1}{h_1} H_0(x) + \left( \frac{1}{h_1} - \frac{1}{h_2} \right) H_1(x) - \theta_2 H_2(x),
\]

\[
\phi_i(x) = L_i(x) + \eta_{i-1} H_{i-1}(x) + \left( \frac{1}{h_i} - \frac{1}{h_{i-1}} \right) H_i(x) - \theta_{i-1} H_{i-1}(x), \quad i = 2, 3, \ldots, J - 2,
\]

\[
\phi_{J-1}(x) = L_{J-1}(x) + \eta_{J-2} H_{J-2}(x) + \left( \frac{1}{h_{J-1}} - \frac{1}{h_J} \right) H_{J-1}(x) - \frac{1}{h_J} H_J(x),
\]

\[
\phi_J(x) = L_J(x) + \eta_{J-1} H_{J-1}(x) + \frac{1}{h_J} H_J(x).
\]

Where

\[
L_0(x) = \begin{cases} 
\frac{1}{h_1} (x - x_1)^2 (x_1 + 3x_0 - 4x), & x \in I_1, \\
0, & \text{otherwise},
\end{cases}
\]

\[
L_1(x) = \begin{cases} 
1 - \frac{1}{h_1} (x - x_1)^2 (x_1 + 3x_0 - 4x), & x \in I_1, \\
\frac{1}{h_2} (x - x_2)^2 (2x + x_2 - 3x_1), & x \in I_2, \\
0, & \text{otherwise},
\end{cases}
\]

\[
L_i(x) = \begin{cases} 
\frac{1}{h_i} (x - x_{i-1})^2 (3x_i - x_{i-1} - 2x), & x \in I_i, \\
\frac{1}{h_{i+1}} (x - x_{i+1})^2 (x_{i+1} - 3x_i + 2x), & x \in I_{i+1}, \\
0, & \text{otherwise},
\end{cases} \quad i = 2, 3, \ldots, J - 2,
\]

\[
L_J(x) = \begin{cases} 
0, & \text{otherwise},
\end{cases}
\]
\[ L_{j-1}(x) = \begin{cases} \frac{1}{h_j} (x - x_{j-1})^2 (3x_{j-1} - x_{j-2} - 2x), & x \in I_{j-1}, \\ 1 - \frac{1}{h_j} (x - x_{j-1})^2 (4x - 3x_j - x_{j-1}), & x \in I_j, \\ 0, & \text{otherwise}, \end{cases} \]

\[ L_j(x) = \begin{cases} \frac{1}{h_j} (x - x_{j-1})^2 (4x - 3x_j - x_{j-1}), & x \in I_j, \\ 0, & \text{otherwise}, \end{cases} \]

\[ H_0(x) = \begin{cases} -\frac{4}{h_1} (x - x_0)(x - x_1)^2, & x \in I_1, \\ 0, & \text{otherwise}, \end{cases} \]

\[ H_1(x) = \begin{cases} \frac{1}{h_1} (x - x_1)(x - x_0), & x \in I_1, \\ \frac{1}{h_j} (x - x_1)(x - x_2)^2, & x \in I_2, \\ 0, & \text{otherwise}, \end{cases} \]

\[ H_i(x) = \begin{cases} \frac{1}{h_i} (x - x_i)(x - x_{i-1})^2, & x \in I_i, \\ \frac{1}{h_{i-1}} (x - x_i)(x - x_{i+1})^2, & x \in I_{i+1}, \\ 0, & \text{otherwise}, \end{cases} \]

\[ i = 2, 3, \ldots, J - 2, \]

\[ H_{j-1}(x) = \begin{cases} \frac{1}{h_{j-1}} (x - x_{j-1})(x - x_{j-2})^2, & x \in I_{j-1}, \\ -\frac{1}{h_j} (x - x_{j-1})(x - x_j), & x \in I_j, \\ 0, & \text{otherwise}, \end{cases} \]

\[ H_j(x) = \begin{cases} -\frac{4}{h_j} (x - x_j)(x - x_{j-1})^2, & x \in I_j, \\ 0, & \text{otherwise}. \end{cases} \]

Let \( V_h = \text{span}\{\phi_1, \phi_2, \ldots, \phi_{j-1}\} \). One can easily see that \( V_h \subset H_0^2(I) \) and \( \phi_i(x_j) = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker delta.

If we take \( j \) elements with equal length \( h \), i.e., \( h_i = h \) for \( i = 1, \ldots, J \), then the basis functions above can be simplified as follows:

\[ \phi_1(x) = \begin{cases} \frac{1}{h_j} (3x_1 - x_0 - 2x)(x - x_0)^2, & x \in I_1, \\ \frac{1}{h_j} (x - x_0)(x - x_2)(2x + x_2 - 3x_1) - \frac{1}{2} (x - x_1)^2, & x \in I_2, \\ -\frac{1}{h_j} (x - x_2)(x - x_3)^2, & x \in I_3, \\ 0, & \text{otherwise}, \end{cases} \]

\[ \phi_j(x) = \begin{cases} \frac{1}{h_j} (x - x_{j-1})(x - x_{j-2})^2, & x \in I_{j-1}, \\ \frac{1}{h_j} (x - x_{j-1})^2 (3x_j - x_{j-1} - 2x) + \frac{1}{2h} (x - x_{j-1})(x - x_j)^2, & x \in I_j, \\ \frac{1}{h_j} (x - x_{j-1})^2 (x_j - 3x_j + 2x) - \frac{1}{2h} (x - x_{j+1})(x - x_j)^2, & x \in I_{j+1}, \\ -\frac{1}{h_j} (x - x_{j+1})(x - x_{j+2})^2, & x \in I_{j+2}, \\ 0, & \text{otherwise}. \end{cases} \]

for \( j = 2, 3, \ldots, J - 2, \)

\[ \phi_{j-1}(x) = \begin{cases} \frac{1}{h_j} (x - x_{j-1})(x - x_{j-2})^2, & x \in I_{j-1}, \\ \frac{1}{h_j} (x - x_{j-1})^2 (3x_j - x_{j-1} - 2x) + \frac{1}{2h} (x - x_{j-1})(x - x_{j-2})^2, & x \in I_{j-1}, \\ \frac{1}{h_j} (2x + x_j - 3x_{j-1})(x - x_j)^2, & x \in I_j, \\ 0, & \text{otherwise}. \end{cases} \]
Remark. The basis functions above can be constructed by replacing the derivatives \( u_x(x_j) \) with \( (u(x_j) - u(x_{j-1}))/2h \) in piecewise cubic Hermite interpolation formula.

Let \( \mathcal{J}_h \) denote the interpolation operator onto \( V_h \). Then we have

\[
\mathcal{J}_h u(x) = \sum_{i=1}^{l-1} u(x_i) \phi_i(x),
\]

and we can show the following approximation property:

\[
\|u - \mathcal{J}_h u\|_{W_p^r} \leq C h^{3-r} \|u\|_{W_p^r}, \quad r = 0, 1, 2, \quad 1 \leq p \leq \infty,
\]

where \( \|\cdot\|_{W_p^r} \) denotes the Sobolev norm.

Let \( \tau \) denote the time step, \( t^n = n \tau \), \( u^n(x) = u(x, t^n) \), \( u^{n+1/2} = (u^{n+1} + u^n)/2 \), and let \( \partial_t u^{n+1} = (u^{n+1} - u^n)/\tau \) for \( n = 0, 1, \ldots, N = T/\tau \). By using the Crank-Nicolson approximation in time discretization, the Galerkin approximation of problem (4) can be defined in the following form:

Find \( U^{n+1} \in V_h \) for \( n = 0, 1, \ldots, N - 1 \) such that

\[
(\partial_t U^{n+1}, w) - \frac{\epsilon}{2} \left((U^{n+1/2})^2, w\right) - \mu(U^{n+1/2}, w) = 0 \quad \forall \, w \in V_h,
\]

where \( \mathcal{J}_h \) denotes differentiation with respect to \( x \). The above algebraic system can be written as the following matrix equation:

\[
\left[2A - \tau \left(\frac{\epsilon}{2} \mathbb{B}^a + \mu C\right)\right] \mathbf{u}_h^{n+1} = \left[2A + \tau \left(\frac{\epsilon}{2} \mathbb{B}^a + \mu C\right)\right] \mathbf{u}_h^n,
\]

where \( \mathbf{u}_h^n = (U_{0h}^n, \ldots, U_{Jh}^n)^T \), and the entries of the \((J - 1) \times (J - 1)\) matrices \( A, \mathbb{B}^a \), and \( C \) are defined by

\[
a_{ij} = (\phi_j, \phi_i), \quad b_{ij}^a = \sum_{k=1}^{J-1} U_{kh}^{n+1/2}(\phi_k \phi_j, \phi_i), \quad c_{ij} = (\phi_j, \phi_i).
\]

The matrix \( A \) is symmetric positive definite, \( C \) is skew-symmetric, and all of the matrices \( A, \mathbb{B}^a \), and \( C \) are septa-diagonal, but the algebraic system is nonlinear. By using a Taylor expansion, one can easily see that for smooth functions \( f(t) \) we get

\[
f(t^{n+1/2}) = \frac{3}{2} f(t^n) - \frac{1}{2} f(t^{n-1}) + O(\tau^2).
\]

We will use an extrapolation \((3U^n - 2U^{n-1})/2\) to approximate \( U^{n+1/2} \) for computing the components \( b_{ij}^a \) of the matrix \( \mathbb{B}^a \). We then obtain the following linearization technique for implementation:

(I) For \( n = 0 \), we use the following refinement to obtain \( \mathbf{U}^1 \):

\[
\left(\frac{\bar{U}^1 - U^0}{\tau}, w\right) - \frac{\epsilon}{2} \left(U^0 \frac{\bar{U}^1 + U^0}{2}, w\right) - \mu \left(U^0_{xx}, w\right) = 0,
\]

\[
(\partial_t \mathbf{U}^1, w) - \frac{\epsilon}{2} \left(\frac{\bar{U}^1 + U^0}{2}, w\right) - \mu \left(U^0_{xx}, w\right) = 0.
\]

(II) For \( n \geq 1 \), we use the following linearized equation instead of (7):

\[
(\partial_t U^{n+1}, w) - \frac{\epsilon}{2} \left(\tilde{\mathbb{E}}_U^n U^{n+1/2}, w\right) - \mu (U^{n+1/2}, w) = 0,
\]

where \( \tilde{\mathbb{E}}_U^n = (3U^n - U^{n-1})/2 \).

Eq. (8) can be written as the linear matrix equation

\[
\left[2A - \tau \left(\frac{\epsilon}{2} \tilde{\mathbb{B}}^a + \mu C\right)\right] \mathbf{u}_h^{n+1} = \left[2A + \tau \left(\frac{\epsilon}{2} \tilde{\mathbb{B}}^a + \mu C\right)\right] \mathbf{u}_h^n,
\]

where \( \tilde{\mathbb{B}}^a \) is the approximation of \( \mathbb{B}^a \) that is obtained by replacing \( U^{n+1/2} \) with \( \tilde{\mathbb{E}}_U^n \).
3. Stability analysis

**Theorem 3.1.** Let $U^n$ be the solution of scheme (7). Then we obtain

$$
\|U^n\|_{L_2} = \|U^0\|_{L_2}, \quad 1 \leq n \leq N.
$$

**Proof.** Taking $w = U^{n+1/2}$ in (7), we have

$$
(\partial_t U^{n+1}, U^{n+1/2}) - \frac{\epsilon}{2} \left( (U^{n+1/2})^2, U^{n+1/2}_x \right) - \mu \left( U^{n+1/2}_{xx}, U^{n+1/2}_x \right) = 0.
$$

By using the boundary conditions, we obtain

$$
(U^{n+1/2})^2, U^{n+1/2}_x = 0.
$$

Then it follows from (10) that

$$
\|U^{n+1}\|_{L_2}^2 = \|U^n\|_{L_2}^2.
$$

This completes the proof. \[
\]

4. Numerical experiments and discussions

To demonstrate the efficiency and accuracy of the present method, we consider several examples in this section. The accuracy is measured by the percentage error (PE), which is defined by

$$
\text{PE} = \frac{|\text{Exact solution} - \text{Approximate solution}|}{|\text{Exact solution}|} \times 100,
$$

and the discrete $L_2$- and $L_\infty$- norms given by

$$
\|u^n - U^n\|_{L_2} = \left( \sum_{j=0}^{J} |u^n_j - U^n_j|^2 \right)^{1/2}, \quad \|u^n - U^n\|_{L_\infty} = \max_{0 \leq j \leq J} |u^n_j - U^n_j|.
$$

The solution of the KdV equation has been found to satisfy many conservation laws. The conservation properties will be examined by calculating the three invariants corresponding to conservation of mass, momentum, and energy [13], viz.,

$$
C_1 = \int_a^b u^n \, dx \simeq h \sum_{j=0}^{J} U^n_j,
$$
$$
C_2 = \int_a^b (u^n)^2 \, dx \simeq h \sum_{j=0}^{J} (U^n_j)^2,
$$
$$
C_3 = \int_a^b \left[ (u^n)^3 - \frac{3\mu}{\epsilon} (u^n)_x^2 \right] \, dx \simeq h \sum_{j=0}^{J} \left[ (U^n_j)^3 - \frac{3\mu}{\epsilon} ((U_j^n)_x)^2 \right],
$$

where

$$
(U_j^n)_x = \sum_{i=1}^{j-1} U^n_i \phi_i'(x_j) = \begin{cases} 
(U_{j+1}^n - U_{j-1}^n)/(2h), & 1 \leq j \leq J - 1, \\
0, & j = 0, J.
\end{cases}
$$

4.1. Numerical simulations for the KdV equation with small parameter $\mu$

In this section, we consider the numerical solutions of the KdV equation with small parameter $\mu$. In order to compare the numerical results obtained by the present method with the ones given in [10,12,22], computations are conducted with parameters $\mu = 4.84 \times 10^{-4}$ and $\epsilon = 1$. The motions of single and two solitary waves are simulated in Examples 1 and 2, respectively.

**Example 1.** The KdV equation has an analytical solution of the form

$$
\frac{d^2 x}{d t} = 3c \sech^2(kx - \omega t - x_0),
$$

(11)
Table 1
Invariants and errors of Example 1 at different times with \( h = 0.005 \) and \( \tau = 0.001 \) on region \( 0 \leq x \leq 2 \).

<table>
<thead>
<tr>
<th>Time</th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( L_2 \times 10^6 )</th>
<th>( L_\infty \times 10^6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.14460</td>
<td>0.08676</td>
<td>0.04685</td>
<td>2.4017</td>
<td>2.1308</td>
</tr>
<tr>
<td>0.005</td>
<td>0.14460</td>
<td>0.08676</td>
<td>0.04691</td>
<td>6.8255</td>
<td>4.5317</td>
</tr>
<tr>
<td>0.01</td>
<td>0.14460</td>
<td>0.08676</td>
<td>0.04691</td>
<td>2.4017</td>
<td>2.1308</td>
</tr>
</tbody>
</table>

Table 2
\( L_2 \)- and \( L_\infty \)-errors and convergence orders for Example 1 at \( t = 0.01 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \tau )</th>
<th>( L_2 \times 10^3 )</th>
<th>Order</th>
<th>( L_\infty \times 10^3 )</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.002</td>
<td>2.6387</td>
<td></td>
<td>8.5078</td>
<td></td>
</tr>
<tr>
<td>0.025</td>
<td>0.001</td>
<td>0.5157</td>
<td>2.3553</td>
<td>2.0661</td>
<td>2.0419</td>
</tr>
<tr>
<td>0.0125</td>
<td>0.0005</td>
<td>0.0439</td>
<td>3.5533</td>
<td>0.1924</td>
<td>2.3739</td>
</tr>
<tr>
<td>0.00625</td>
<td>0.00025</td>
<td>0.0039</td>
<td>3.4890</td>
<td>0.0205</td>
<td>3.2284</td>
</tr>
</tbody>
</table>

Table 3
PEs of Example 1 at \( t = 0.005 \) on region \( 0 \leq x \leq 2 \) for selected values of \( x \) with different mesh sizes.

<table>
<thead>
<tr>
<th>Method</th>
<th>( h )</th>
<th>( \tau )</th>
<th>( x = 0.2 )</th>
<th>( x = 0.4 )</th>
<th>( x = 0.6 )</th>
<th>( x = 0.8 )</th>
<th>( x = 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>0.02</td>
<td>0.005/3</td>
<td>0.0021</td>
<td>0.0599</td>
<td>0.0029</td>
<td>0.0007</td>
<td>0.0080</td>
</tr>
<tr>
<td></td>
<td>0.0125</td>
<td>0.005/8</td>
<td>0.0023</td>
<td>0.0597</td>
<td>0.0030</td>
<td>0.0007</td>
<td>0.0081</td>
</tr>
<tr>
<td></td>
<td>0.02</td>
<td>0.005/5</td>
<td>0.0026</td>
<td>0.0110</td>
<td>0.0009</td>
<td>0.0010</td>
<td>0.0012</td>
</tr>
<tr>
<td></td>
<td>0.0125</td>
<td>0.005/5</td>
<td>0.0022</td>
<td>0.0597</td>
<td>0.0030</td>
<td>0.0007</td>
<td>0.0081</td>
</tr>
<tr>
<td>[22]</td>
<td>0.02</td>
<td>0.005/5</td>
<td>0.0271</td>
<td>0.1105</td>
<td>0.0216</td>
<td>0.0470</td>
<td>0.1067</td>
</tr>
<tr>
<td>[10]</td>
<td>0.0125</td>
<td>0.005/5</td>
<td>0.0017</td>
<td>0.0109</td>
<td>0.0009</td>
<td>0.0010</td>
<td>0.0012</td>
</tr>
<tr>
<td>[12]</td>
<td>0.005/3</td>
<td>0.0003</td>
<td>0.0399</td>
<td>0.0175</td>
<td>0.0007</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>0.005/8</td>
<td>0.0003</td>
<td>0.0057</td>
<td>0.0022</td>
<td>0.0007</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Table 4
PEs of Example 1 at \( t = 0.01 \) on region \( 0 \leq x \leq 2 \) for selected values of \( x \) with different mesh sizes.

<table>
<thead>
<tr>
<th>Method</th>
<th>( h )</th>
<th>( \tau )</th>
<th>( x = 0.2 )</th>
<th>( x = 0.4 )</th>
<th>( x = 0.6 )</th>
<th>( x = 0.8 )</th>
<th>( x = 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>0.02</td>
<td>0.01/6</td>
<td>0.0166</td>
<td>0.1579</td>
<td>0.0011</td>
<td>0.0116</td>
<td>0.0119</td>
</tr>
<tr>
<td></td>
<td>0.0125</td>
<td>0.01/6</td>
<td>0.0130</td>
<td>0.0329</td>
<td>0.0006</td>
<td>0.0027</td>
<td>0.0072</td>
</tr>
<tr>
<td></td>
<td>0.02</td>
<td>0.01/10</td>
<td>0.0171</td>
<td>0.1582</td>
<td>0.0010</td>
<td>0.0112</td>
<td>0.0127</td>
</tr>
<tr>
<td></td>
<td>0.0125</td>
<td>0.01/10</td>
<td>0.0080</td>
<td>0.0328</td>
<td>0.0007</td>
<td>0.0022</td>
<td>0.0027</td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>0.01/6</td>
<td>0.0122</td>
<td>0.0147</td>
<td>0.0001</td>
<td>0.0251</td>
<td>0.4107</td>
</tr>
<tr>
<td>[22]</td>
<td>0.02</td>
<td>0.01/10</td>
<td>0.0701</td>
<td>0.2344</td>
<td>0.0030</td>
<td>0.0984</td>
<td>0.1029</td>
</tr>
<tr>
<td>[10]</td>
<td>0.0125</td>
<td>0.01/10</td>
<td>0.0024</td>
<td>0.0042</td>
<td>0.0195</td>
<td>0.0240</td>
<td>0.0000</td>
</tr>
<tr>
<td>[12]</td>
<td>0.01/6</td>
<td>0.0003</td>
<td>0.1648</td>
<td>0.0663</td>
<td>0.0057</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>0.01/16</td>
<td>0.0067</td>
<td>0.0242</td>
<td>0.0072</td>
<td>0.0063</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

where

\[
\kappa = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} \quad \text{and} \quad \omega = \varepsilon \kappa.
\]

It presents a single solitary wave with amplitude of 3c. In our simulation, the initial condition is obtained from (11) by taking \( t = 0 \), \( c = 0.3 \), and \( x_0 = 6 \). The analytic values of the three invariants are \( C_1 = 0.14460, C_2 = 0.08676, \) and \( C_3 = 0.04685 \), respectively. The errors in \( L_2 \)- and \( L_\infty \)-norms and the numerical values of the three invariants are given in Table 1. The quantities of the invariants remain almost constant while the simulation runs up to \( t = 0.01 \). The numerical invariant values of \( C_1, C_2, \) and \( C_3 \) are changed by less than \( 8.1350 \times 10^{-4}\% \), \( 4.6917 \times 10^{-4}\% \), and \( 0.12\% \) relative to the analytic values up to time \( t = 0.01 \), respectively. Table 2 shows the convergence rate of the present method at \( t = 0.01 \) in \( L_2 \)- and \( L_\infty \)-norms.

For a comparison, we compute PEs at some selected values of space variable \( x \) using different mesh sizes of \( \tau \) and \( h \), and we record the results as well as the ones given by [10,12,22] in Tables 3 and 4. One can see from these tables that the numerical solutions obtained by the present method are more accurate than the ones shown in [22], and are comparable to...
the ones in [10,12]. Fig. 1 shows the numerical and exact solutions, and absolute error distribution at time $t = 0.01$, from which one can see that the maximum error occurs around the peak position of wave amplitude.

In order to see the long time behavior of a single solitary wave, we also give a simulation on bigger time domain $[0,3]$. The numerical and exact solution curves at different times are shown in Fig. 2. One can see that the numerical and exact solutions are in good agreement.

**Example 2.** We consider the motion of two solitary waves. The initial condition is obtained from the analytic solution (see [4])

$$u(x, t) = 12 \mu (\log F)_{xx}, \quad 0 \leq x \leq 4,$$

(12)

where

$$F = 1 + e^{\eta_1} + e^{\eta_2} + \left( \frac{x - \alpha_2}{x_1 + \alpha_2} \right) e^{\eta_1 + \eta_2},$$

$$\eta_i = \alpha_i \mu + b_i, \quad i = 1, 2,$$

$$\alpha_1 = \sqrt{\frac{0.3}{\mu}}, \quad \alpha_2 = \sqrt{\frac{0.1}{\mu}}, \quad b_1 = -0.48 \alpha_1, \quad \text{and} \quad b_2 = -1.07 \alpha_2.$$

The analytic values of the three invariants are $C_1 = 0.0991$, $C_2 = 0.0195$, and $C_3 = 0.0024$. The errors in the discrete $L_2$- and $L_\infty$-norms and the numerical values of $C_1$, $C_2$, and $C_3$ at $t = 0.005$ and 0.01 are shown in Table 5. As shown in this table, the quantities of the invariants remain almost constantly while the simulation runs up to $t = 0.01$. The numerical invariant
values of $C_1$, $C_2$, and $C_3$ are changed by less than 1.6277 × 10^{-4}\%$, 2.5948 × 10^{-5}\%, and 1.8063 × 10^{-3}\% relative to the analytic invariant values up to time $t = 0.01$, respectively.

We also record PEs at some selected values of $x$ using different mesh sizes and the ones of other available methods in Tables 6 and 7 at $t = 0.005$ and 0.01, respectively. The PEs are almost same as the ones given in [10,12]. Fig. 3 shows the numerical and exact solution curves, and they are in good agreement. Fig. 4 displays the numerical and exact solutions and the distribution of absolute errors at time $t = 0.01$.
4.2. Numerical simulations for the KdV equation on bigger time domains

In this section, we consider the motions of single solitary waves and the interactions of several solitary waves on bigger time domains. All numerical experiments are conducted with the parameters \( \varepsilon = \mu = 1 \).

**Example 3.** We consider the motion of a single solitary wave and solve \((1)\) on region \([-40, 100]\) from time \( t = 0 \) to 200. We use \((11)\) as the analytical solution of \((1)\). By setting \( \kappa = 0.3, \ x_0 = 0, \ c = 4\kappa^2, \) and \( \omega = 4\kappa^3, \) we obtain the initial condition

\[
u_0(x) = 12\kappa^2 \text{sech}^2 \kappa x.
\]

The experiments are done with \( h = 1/3 \) and \( \tau = 1/3 \). \textbf{Table 8} shows the numerical invariants and errors in the discrete \( L_2 \)- and \( L_\infty \)-norms. As we may expect, the numerical invariant values remain almost constantly while the simulation runs up to \( t = 200 \). The relative errors of numerical invariant values of \( C_1, \ C_2, \) and \( C_3 \) are less than 0.12\%, 0.27\%, and 0.14\%, respectively.

\[
\begin{array}{cccccc}
\text{Time} & C_1 & C_2 & C_3 & L_2 & L_\infty \\
\hline
0 & 7.2000 & 5.1840 & 3.3592 & & \\
40 & 7.1982 & 5.1812 & 3.3668 & 0.0070 & 0.0032 \\
80 & 7.1972 & 5.1783 & 3.3637 & 0.0174 & 0.0077 \\
120 & 7.1947 & 5.1755 & 3.3606 & 0.0311 & 0.0134 \\
160 & 7.1927 & 5.1727 & 3.3576 & 0.0481 & 0.0206 \\
200 & 7.1911 & 5.1698 & 3.3545 & 0.0681 & 0.0288
\end{array}
\]

**Fig. 4.** Numerical and exact solutions (left) and absolute error distribution (right) of \textbf{Example 2} at \( t = 0.01 \) with \( h = 0.0125 \) and \( \tau = 0.001 \).

**Fig. 5.** Profiles of numerical (dots) and exact (lines) solutions of \textbf{Example 3} for the motion of single solitary wave with \( h = 1/3 \) and \( \tau = 1/3 \).
Fig. 5 shows the numerical and exact solitary waves, and they are in good agreement. The numerical and exact solutions, and the distribution of absolute errors at time $t = 200$ are shown in Fig. 6, and one can see that the maximum error occurs around the peak position of wave amplitude.

**Example 4.** We consider the interaction of three solitary waves and solve the KdV equation numerically on region $[-90,90]$ from $t = 0$ to $280$ with $h = 1/3$ and $\tau = 1/3$. The initial condition is given by (see [15])

$$u(x,0) = \sum_{i=1}^{3} 12\kappa_i^3 + \text{sech}^2[\kappa_i(x-x_i)], \quad -90 \leq x \leq 90,$$

where

$$\kappa_1 = 0.3, \quad \kappa_2 = 0.25, \quad \kappa_3 = 0.2, \quad x_1 = -60, \quad x_2 = -44, \quad \text{and} \quad x_3 = -26.$$
Table 9
Invariants of Example 4 with \( \tau = 1/3 \) and \( h = 1/3 \).

<table>
<thead>
<tr>
<th>Time</th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>18.0000</td>
<td>9.8274</td>
<td>5.2623</td>
</tr>
<tr>
<td>40</td>
<td>18.0010</td>
<td>9.8273</td>
<td>5.2732</td>
</tr>
<tr>
<td>80</td>
<td>18.0057</td>
<td>9.8297</td>
<td>5.2699</td>
</tr>
<tr>
<td>120</td>
<td>18.0039</td>
<td>9.8275</td>
<td>5.2707</td>
</tr>
<tr>
<td>160</td>
<td>17.9990</td>
<td>9.8218</td>
<td>5.2703</td>
</tr>
<tr>
<td>200</td>
<td>17.9949</td>
<td>9.8181</td>
<td>5.2671</td>
</tr>
</tbody>
</table>

Table 10
Invariants of Example 5 with \( \tau = 0.5 \) and \( h = 0.5 \).

<table>
<thead>
<tr>
<th>Time</th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>21.6000</td>
<td>10.3887</td>
<td>5.2688</td>
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<tr>
<td>40</td>
<td>21.5829</td>
<td>10.3697</td>
<td>5.2794</td>
</tr>
<tr>
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<td>21.5993</td>
<td>10.3733</td>
<td>5.2580</td>
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<tr>
<td>120</td>
<td>21.5945</td>
<td>10.3664</td>
<td>5.2597</td>
</tr>
<tr>
<td>160</td>
<td>21.5744</td>
<td>10.3407</td>
<td>5.2494</td>
</tr>
<tr>
<td>200</td>
<td>21.5602</td>
<td>10.3208</td>
<td>5.2292</td>
</tr>
</tbody>
</table>

Fig. 8. Interaction of three solitary waves of Example 4 with \( \tau = 1/3 \) and \( h = 1/3 \).

Fig. 9. Interaction of four solitary waves of Example 5 with \( \tau = 0.5 \) and \( h = 0.5 \) at different times.
Fig. 7 shows the development of the interaction of three solitons at different times, and Fig. 8 displays the interaction globally. From these figures, one can see that our method is stable and the results accord with the ones given in [15]. Meanwhile, the numerical invariant values are given in Table 9, from which one can see that the numerical solutions satisfy the three conservation laws. The relative errors of computed invariant values of $C_1$, $C_2$, and $C_3$ are less than $2.8094 \times 10^{-2}$%, $9.5498 \times 10^{-2}$%, and $9.1584 \times 10^{-2}$%, respectively.

Example 5. We consider the interaction of four solitary waves in this example. The computation are done on region $[-120, 120]$ from $t = 0$ to $400$, with $h = 0.5$ and $\tau = 0.5$. The initial condition is given by (see [15])

$$u(x, 0) = \sum_{i=1}^{4} 12\kappa_i^2 \sech^2[\kappa_i(x - x_i)], \quad -120 \leq x \leq 120,$$

where

$$\kappa_1 = 0.3, \quad \kappa_2 = 0.25, \quad \kappa_3 = 0.2, \quad \kappa_4 = 0.15,$$

$$x_1 = -85, \quad x_2 = -60, \quad x_3 = -35, \quad \text{and} \quad x_4 = -10.$$

We record the numerical invariant values in Table 10, and one can see that the numerical solutions satisfy the three conservation laws. The relative errors of the numerical invariant values of $C_1$, $C_2$, and $C_3$ are less than $0.18\%$, $0.65\%$, and $0.75\%$, respectively. The development of the interaction of four solitons is shown in Figs. 9 and 10 shows a global view of the interaction. These figures show that the present method is stable and gives a good simulation for the interaction of solitary waves. These results accord with the ones given in [15].

5. Conclusion

The KdV equation was numerically solved by the Galerkin approach using a new basis of smooth piecewise cubic polynomials. The performance of the present method was evaluated in terms of the percentage error or the discrete $L_2$- and $L_\infty$-norms. The conservation laws of the solutions are also considered for the present method. Our numerical experiments include the cases of small and larger parameters on small and bigger time domains, respectively. The motions of a single and two solitary waves, the interactions of three and four solitary waves were simulated. Based on our numerical experiments, we may say that the present method gives accurate numerical solutions and is useful for simulations of the conservation properties, motions of solitary waves, and interactions of solitary waves for the KdV equation.

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References


