Estimation of spatially varying open boundary conditions for a numerical internal tidal model with adjoint method

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Abstract

The adjoint data assimilation technique is applied to the estimation of the spatially varying open boundary conditions (OBCs) for a numerical internal tidal model. The spatial variation of the OBCs is realized by the so-called 'independent point scheme' (IPS): a subset is chosen as the independent points from the full set of open boundary points and the OBCs are obtained through linear interpolation of the values at the independent points. A series of ideal experiments are carried out on a real topography to further test this assimilation model, and to numerically investigate some properties of the IPS. On the basis of the numerical results, it is shown that, in most cases, the use of the IPS can indeed effectively improve the precision of the estimation of the OBCs. Furthermore, if the independent points can be arranged reasonably the improvement may be remarkable. The IPS shows us a way to improve the estimation of the OBCs for this model.

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1. Introduction

In a regional ocean model, the open boundary conditions (OBCs), which must be prescribed to complete the model description at non-land boundaries, are very important and have a critical impact on the modeling results. The treatment of OBCs is an old topic and has been a concern in the regional ocean modeling for sometime. Many schemes have been proposed for different situations. Broadly, OBCs can be divided into two classes, passive and active [49]. The passive OBCs generally fall into two categories: the radiation and characteristic boundary conditions and the relaxation (sponge) conditions. The discussion on the passive boundary conditions is still one of the most interesting topics in recent years. On the other hand, the active OBCs are used to drive the simulation, cf. [3,10,30], and references therein. Marchesiello et al. [41] proposed a combination, an adaptive boundary condition, where different definitions are used depending on whether information is entering or leaving the domain. In this paper, the passive Flather condition [25] is modified to an active condition on normal velocity, by including the effect of known values of elevation and normal

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velocity that represent conditions beyond the model boundary (i.e. tidal force), which has also been used in quite a few studies (e.g. [10,30]).

In the active OBCs, the boundary solution is fully or partially specified by using external data at every boundary point or at boundary inflow points only. In practical ocean modeling, on the other hand, the external data can be obtained either from available observations near the open boundaries (tidal gauge data or satellite data) or, within a nested approach, from larger domain numerical models such as Schwiderski’s global tidal model [56] and the TPXO.3 global tidal model [21]. Unfortunately, observations at open waters are often scarce and the global tidal model results are less accurate in shallow waters. Therefore, a major difficulty faced by regional ocean models is concerned with the treatment of the OBCs [35].

With the development of large ocean observing programs and remote sensing techniques, increasingly more oceanic data is becoming available. This provides a promising prospect for improving the precision of ocean modeling with these observations. In ocean modeling, it is common to solve the inverse problem, in which the OBCs are considered as tunable control parameters of the dynamical model and are determined, at least in part, from measurement data in the interior of the domain by optimization methods. That is, in the general context of control theory, one interchanges the roles of boundary values (no longer known) and the interior fields (partially known from observations). Control theory provides a general theoretical understanding of such problems, and in a fundamental sense, there is no doubt that such methods work very effectively [3]. A few attempts have been made previously (e.g. [43,57]). Indeed, this is an active topic of a fast-growing research field—data assimilation, which is an effective method for marine research, and has become widely used in meteorological and oceanographic predictions in recent years. Among all data assimilation methods, four-dimensional variational (4D-Var) data assimilation is one of the most effective and powerful approaches developed over the past two decades. It is an advanced data assimilation method which involves the adjoint technique using the optimization technique based upon Lagrange multipliers (e.g. [37,66]), and has the advantage of directly assimilating various observations distributed in time and space into the numerical model while maintaining dynamical and physical consistency with the model. Rodrigues et al. [54] applied an adjoint problem formulation to the simultaneous estimation of spatially dependent diffusion coefficient and source term in a one-dimensional nonlinear diffusion problem. In their approach, no a priori assumption is required regarding the functional form of the unknowns. Using an adjoint sensitive method, Tber et al. [61] studied the simultaneous identification of spatially distributed hydraulic conductivity and storativity in a seawater intrusion model. The 4D-Var data assimilation with the adjoint method has also been widely applied to the estimation of the OBCs. Early studies include Lardner et al. [35], Seiler [57], Bennett and McIntosh [8], Bennett [6], Zou et al. [75], Heemink et al. [29]; recent texts are Ayoub [3], Zhang et al. [71], Gejadze and Copeland [26], Gejadze et al. [27], Nguyen [46], Zhang and Lu [72,73], Chen et al. [14].

This study is an extension of [14] (hereafter CML2012), which constructed an adjoint assimilation model for numerical simulation of internal tides. In CML2012, the model was simply tested with a series of ideal experiments, in which several prescribed spatial distributions of OBCs were successfully inverted by assimilating the model-generated pseudo-observations. However, although the inverted OBCs were consistent with the prescribed ones very well, a considerable amount of noises (small fluctuations) are detectable in the solution. This may be caused, to some extent, by the ill-posedness of the optimization problem. As noted by Yeh and Sun [68] and Yeh [67] in the work of ground water flow parameter estimation, the inverse or parameter estimation problem is often ill-posed and beset by instability and non-uniqueness, particularly if one seeks parameters distributed in space and time domain. The same viewpoint has been put forward by Heemink et al. [29] Zhang and Lu [72], Smedstad and O’Brien [60], Das and Lardner [19,20], Lardner and Das [36], Ullman and Wilson [65], Alekseev et al. [1]. In these works they proposed to insert an additional criterion into the cost function named the penalty term, and by doing this, large fluctuations will be penalized to ensure that the parameters vary smoothly. Numerous studies indicate that the problem of primitive equations with OBCs is ill-posed in the sense that the existence of a unique physically realistic solution is not guaranteed (see [7,9,12,32,41,48]). Under some conditions the inviscid primitive equations may be well-posed, e.g. ‘if boundary conditions are formulated in terms of local eigenfunction expansions or nonlocal boundary operators are used’ [48]. Blayo and Debreu [9] pointed out that even well-posed model equations do not guarantee an accurate solution. Lu and Zhang [40], Zhang and Lu [72] and Zhang et al. [74] demonstrated that the use of the independent points to determine the 2-D bottom friction coefficient is effective as well as physically reasonable. Nevertheless, it is still valuable, although simple, to have an examination of the performance of this technique (called ‘independent point scheme’ (IPS) in this paper) when borrowed to the estimation of the spatially varying OBCs in a system where the available observations are sparse and much fewer than those in previous work of CML2012, which is the main objective of this paper.
This article is organized as follows: we start by having a brief introduction of the two-layered internal tidal model and the optimization method in Section 2, and the description of the IPS is also included at the end of this section. For the details of the model refer to CML2012. In Section 3, as illustrative numerical examples, a series of ideal experiments are carried out to invert the prescribed OBCs on a real topography and the IPS is tested. The experimental results are discussed in detail in Section 4, and comparison with a classical regularization method as well as an examination of the suggested open boundary control scheme involving the Flather condition are also made in this section. Finally, we make a summary and draw some conclusions in Section 5.

2. Model and optimization method

2.1. Forward model

A two-layer version of the model described in CML2012 is considered in this paper (see Fig. 1). Assuming the potential density $\rho_k$ ($k = 1, 2$) in each layer is constant, the layer-averaged, nonlinear, time-dependent continuity and momentum equations of each layer subject to the hydrostatic approximations are derived from the primitive 3-D governing equations. Here, the subscripts 1 and 2 refer to the upper and lower layer, respectively. Using spherical coordinates in the horizontal direction and isopycnic coordinates in the vertical, we obtain the internal mode equations as follows.

Upper layer:

$$\frac{\partial q_1}{\partial t} + \frac{1}{R \cos \phi} \frac{\partial (q_1 u_1)}{\partial \lambda} + \frac{1}{R \cos \phi} \frac{\partial (q_1 v_1 \cos \phi)}{\partial \phi} = 0 \quad (1a)$$

$$\frac{\partial u_1}{\partial t} + \frac{u_1}{R \cos \phi} \frac{\partial u_1}{\partial \lambda} + \frac{v_1}{R} \frac{\partial u_1}{\partial \phi} - u_1 v_1 \tan \phi - f v_1 + \frac{g}{R \cos \phi} \frac{\partial}{\partial \phi} \sum_{m=1}^{l} \left( \frac{q_m}{\rho_m} - h_m \right) A \Delta u_1 + A \frac{\bar{\rho}}{q_1 h} (u_1 - u_2) = 0 \quad (1b)$$

Fig. 1. Definition sketch for a two-layered flow.
Here, the Lower \( \rho \) = \( M_\lambda U \Delta \partial V - \partial t + 2 \bar{\text{barotropic}} \)  

\[ \frac{\partial v_1}{\partial t} + \frac{u_1}{R \cos \phi} \frac{\partial v_1}{\partial \lambda} + \frac{v_1}{R} \frac{\partial v_1}{\partial \phi} + \frac{u_1^2 \tan \phi}{R} + fu_1 + \frac{g}{R} \frac{1}{\rho_m} \sum_{m=1}^{\infty} \left( \frac{q_m}{\rho_m} - h_m \right) \]

\[ - A_h \Delta v_1 + \frac{A_v \bar{\rho}}{q_1 h} (v_1 - v_2) = 0 \]  

(1c)

Lower layer:

\[ \frac{\partial q_2}{\partial t} + \frac{1}{R \cos \phi} \frac{\partial (q_2 u_2)}{\partial \lambda} + \frac{1}{R \cos \phi} \frac{\partial (q_2 v_2 \cos \phi)}{\partial \phi} = 0 \]  

(2a)

\[ \frac{\partial u_2}{\partial t} + \frac{u_2}{R \cos \phi} \frac{\partial u_2}{\partial \lambda} + \frac{v_2}{R} \frac{\partial u_2}{\partial \phi} - \frac{u_2 v_2 \tan \phi}{R} - fu_2 + \frac{g}{R \cos \phi} \frac{1}{\rho_2} \sum_{k=1}^{\infty} \left( \frac{q_k}{\rho_2} - h_k \right) - A_h \Delta u_2 \]

\[ - \frac{A_v \bar{\rho}}{q_2 h} (u_1 - u_2) + \frac{\kappa^2 u_2}{q_2} \sqrt{(u_2)^2 + (v_2)^2} = 0 \]  

(2b)

\[ \frac{\partial v_2}{\partial t} + \frac{u_2}{R \cos \phi} \frac{\partial v_2}{\partial \lambda} + \frac{v_2}{R} \frac{\partial v_2}{\partial \phi} + \frac{u_2^2 \tan \phi}{R} + fu_2 + \frac{g}{R \cos \phi} \frac{1}{\rho_2} \sum_{k=1}^{\infty} \left( \frac{q_k}{\rho_2} - h_k \right) - A_h \Delta v_2 - \frac{A_v \bar{\rho}}{q_2 h} (v_1 - v_2) \]

\[ + \frac{\kappa^2 v_2}{q_2} \sqrt{(u_2)^2 + (v_2)^2} = 0 \]  

(2c)

Here, \( t \) is the time, \( \lambda \) and \( \phi \) are the east longitude and north latitude, respectively, \( u(\lambda, \phi, t) \) and \( v(\lambda, \phi, t) \) are horizontal velocities in \( \lambda \) and \( \phi \), respectively, \( q(\lambda, \phi, t) \) is the time-varying layer mass and \( q_1 = \rho_1 (h_1 + \eta_1 - \eta_2) \), \( q_2 = \rho_2 (h_1 + \eta_2) \) where \( h_k \) and \( \eta_k \) are the undisturbed layer thickness and interface (surface for \( k = 1 \)) elevation above the undisturbed level, respectively. \( R \) is the radius of the earth, \( g \) the gravitational acceleration, \( f \) the Coriolis parameter and \( f = 2 \Omega \sin \phi \), where \( \Omega \) represents the angular speed of Earth’s rotation, \( A_h \) the horizontal eddy viscosity coefficient, \( \Delta \) the Laplace operator and

\[ \Delta(u, v) = \frac{1}{R^2 \cos^2 \phi} \frac{\partial^2 (u, v)}{\partial \lambda^2} + \frac{1}{R^2 \cos \phi} \frac{\partial}{\partial \phi} \left[ \cos \phi \frac{\partial (u, v)}{\partial \phi} \right] \]

\( A_v \) and \( \kappa \) are the interface and bottom friction coefficient, respectively, \( \bar{\rho} = (\rho_1 + \rho_2)/2 \) and \( \bar{h} = (h_1 + h_2)/2 \). In the forward model \( q, u \) and \( v \) are the main output and called the state variables in this paper.

The barotropic currents are defined by

\[ U = \frac{q_1 u_1 + q_2 u_2}{Q}, \quad V = \frac{q_1 v_1 + q_2 v_2}{Q}, \quad \bar{V} = \frac{q_1 v_1 + q_2 v_2}{Q} \]

where \( Q = q_1 + q_2 \).

Integrating the continuity equations by Eq. (1a) + Eq. (2a) and integrating the momentum equations by \( q_1 \) Eq. (1b) + \( q_2 \) Eq. (2b) and \( q_1 \) Eq. (1c) + \( q_2 \) Eq. (2c), we get the 2-D external mode as follows:

\[ \frac{\partial Q}{\partial t} + \frac{1}{R \cos \phi} \frac{\partial (Q U)}{\partial \lambda} + \frac{1}{R \cos \phi} \frac{\partial (Q V \cos \phi)}{\partial \phi} = 0 \]  

(3a)

\[ \frac{\partial U}{\partial t} - fV + M_\lambda + R_\phi + \frac{\kappa^2 u_2}{Q} \sqrt{(u_2)^2 + (v_2)^2} = 0 \]  

(3b)

\[ \frac{\partial V}{\partial t} + fU + M_\phi + R_\lambda + \frac{\kappa^2 v_2}{Q} \sqrt{(u_2)^2 + (v_2)^2} = 0 \]  

(3c)

where

\[ M_\lambda = \frac{g}{R \cos \phi} \frac{1}{\rho_2} \frac{\partial}{\partial \lambda} \left( \frac{Q}{\rho} - h \right) + \frac{gq_2}{Q R \cos \phi} \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \frac{\partial q_1}{\partial \lambda}, \]
\[ M_\phi = \frac{g}{R} \frac{\partial}{\partial \phi} \left( \frac{Q}{\rho} - h \right) + \frac{gq_2}{Q} \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \frac{\partial q_1}{\partial \phi}, \]

\[ R_U = \frac{1}{Q} \sum_{k=1}^{2} q_k \left( \frac{u_k}{R \cos \phi} \frac{\partial u_k}{\partial \phi} + \frac{v_k}{R} \frac{\partial v_k}{\partial \phi} - \frac{u_k v_k \tan \phi}{R} \right) - \frac{A_h}{Q} \sum_{k=1}^{2} q_k \Delta u_k - \frac{2}{Q} \sum_{k=1} v_k \frac{\partial}{\partial t} \left( \frac{q_k}{Q} \right), \]

\[ R_V = \frac{1}{Q} \sum_{k=1}^{2} q_k \left( \frac{u_k}{R \cos \phi} \frac{\partial v_k}{\partial \phi} + \frac{v_k}{R} \frac{\partial u_k}{\partial \phi} + \frac{u_k^2 \tan \phi}{R} \right) - \frac{A_h}{Q} \sum_{k=1}^{2} q_k \Delta v_k - \frac{2}{Q} \sum_{k=1} v_k \frac{\partial}{\partial t} \left( \frac{q_k}{Q} \right). \]

Here \( h(\lambda, \phi) = h_1 + h_2 \) is the undisturbed water depth; \( \rho(\lambda, \phi, \theta) \) is the vertically averaged density and \( \rho = Q(h + \eta_1) \). Note that the second terms of \( M_\phi \) and \( M_\phi \) represent the resultant effect of the horizontal density gradients in all layers on the sea surface and are the main causes of the surface manifestation of internal tides.

2.2. Adjoint model

The adjoint method is a powerful tool for parameter estimation. The basic idea of the adjoint method is quite simple: a model is defined by an algorithm and its control parameters including initial conditions, boundary conditions and empirical parameters. The cost function that measures the data misfit between the model output and observations is minimized through optimizing the control parameters. In detail, the cost function decreases along a certain search direction which can be calculated with a certain optimization algorithm according to the gradient of cost function with respect to the control parameters, and this gradient is calculated by what has been known as the adjoint model. Based on the governing Eqs. (1a)–(2c) of the forward model, its adjoint model can be constructed as follows. The details of the adjoint model in a generalized form (not limited to the two-layer case) can be found in the work of CML2012.

The cost function is defined as

\[ J(q, u, v; p) = \frac{1}{2} \int \sum_{\Sigma} \left[ K_u \sum_{k=1}^{2} (u_k - \hat{u}_k)^2 + K_v \sum_{k=1}^{2} (v_k - \hat{v}_k)^2 \right] d\sigma \]  

where \( k \) is layer index, \( \Sigma \) denotes the computing area of both time and space, \( p \) represents the generalized control parameters, \( u_k \) and \( v_k \) are the model simulated, \( \hat{u}_k \) and \( \hat{v}_k \) are the observations. Here \( K_u \) and \( K_v \) are the weight matrices and theoretically should be the inverse of the observation error covariance matrix. That is, the cost function is weighted more heavily toward the observations that are more accurate or important. However, determining the correct forms for \( K_u \) and \( K_v \) is far from easy [2]. In this paper, since only ideal experiments with model-generated pseudo-observations are performed, by assuming that the errors in the data are uncorrelated and equally weighted, \( K_u \) and \( K_v \) are reduced to unit matrices [69].

The Lagrangian function is defined by

\[ L(q, u, v; q_a, u_a, v_a; p) = J(q, u, v; p) + \int \sum_{\Sigma} [q_{a1} \cdot Eq. (1a) + u_{a1} q_1 \cdot Eq. (1b) + v_{a1} q_1 \cdot Eq. (1c)] d\sigma \]

\[ + \int \sum_{\Sigma} [q_{a2} \cdot Eq. (2a) + u_{a2} q_2 \cdot Eq. (2b) + v_{a2} q_2 \cdot Eq. (2c)] d\sigma \]

where \( q_{ak}, u_{ak} \) and \( v_{ak} \) are called adjoint variables of state variables \( q_k, u_k \) and \( v_k \), respectively.

According to the typical theory of Lagrangian multiplier method, we have the following first-order derivates of Lagrangian function with respect to all the variables and parameters:

\[ \frac{\partial L}{\partial q_{ak}} = 0, \quad \frac{\partial L}{\partial u_{ak}} = 0, \quad \frac{\partial L}{\partial v_{ak}} = 0; \]

\[ \frac{\partial L}{\partial q_k} = 0, \quad \frac{\partial L}{\partial u_k} = 0, \quad \frac{\partial L}{\partial v_k} = 0; \]
\[ \frac{\partial L}{\partial p} = 0. \]  

(8)

Eq. (6) returns the governing Eqs. (1a)–(2c). The adjoint equations can be derived from Eq. (7). From Eq. (8) we can obtain the gradients of the cost function with respect to control parameters.

Similar to the forward model, the adjoint model also consists of the internal and external modes. Actually, the equations derived from Eq. (7) are considered as the internal mode and the external mode can be derived from the internal mode in a similar way the external mode of the forward model is derived. The details of both the internal and external modes of the adjoint model can be found in CML2012.

2.3. Discretization

Several numerical methods have been widely used in the discretization of time-dependent 3-D primitive equations. The time integration schemes of these methods can be fully explicit [38], semi-implicit [4,11] or fully implicit [70]. For large-scale oceanic problems, the applications of 3-D models are becoming a reality with the aid of modern computers. The fully explicit finite difference method is relatively simple to implement, except that its time step is strictly restricted by the Courant–Friedrich–Lewy (CFL) stability criterion [53]. At present, many existing ocean models are based on an Alternating Direction Implicit (ADI) method which was proposed for the approximate solution of the shallow-water equations by Fairweather and Navon [24]. ADI method results in computational efficiency superior to fully explicit methods because their improved stability allows large time steps to be employed, and it is also easy for implementation (see [72,73]). Since the model must simulate fields of both velocity and elevations in each isopycnic layer, a technique known as external–internal mode splitting has been used in several ocean models in Simons [59].

In the model formulation, the forward model is split into an external mode and an internal mode. A system of 2-D vertically integrated Eqs. (3a)–(3c) (external mode) is solved independently from the 3-D Eqs. (1a)–(2c) (internal mode). Arakawa C grid is used in the finite difference forms. In both the external and internal modes, the whole-time step \( \Delta T \) is divided into two half-time steps \( \Delta T_1 = \Delta T/2 \) for computation. In each half-time step, the external mode is calculated prior to the calculation of the internal mode. The surface elevations \( (\eta_1) \) and depth-averaged currents \( (U \text{ and } V) \) can be calculated efficiently by the external mode and can be used to adjust the solutions of the internal mode. The ADI method is employed for the external mode computations and the time step of external mode is thus not restricted by the CFL condition. A semi-implicit scheme is used for the internal mode computations, which give the vertical structure of internal tides. The discretization of the adjoint model is similar to that of the forward model. The grid locations of the adjoint variables \( q_a, u_a \) and \( v_a \) are the same as those of the corresponding state variables \( q, u \) and \( v \), respectively. During computation of the adjoint model, the solutions of the internal mode are also adjusted by those of the external mode which is calculated earlier.

The information is transferred from external mode to internal mode through two ways. One is the so-called Montgomery potential, i.e. the terms \( M_1 = g \sum_{m=1}^{l} (q_m / \rho_m - h_m) \) and \( M_2 = g \sum_{k=1}^{2} (q_k / \rho_2 - h_k) \) involved in Eqs. (1b), (1c), (2b) and (2c), respectively. The term \( M_1 \) is contributed by sea surface elevation, which leads to barotropic motions while \( M_2 \) contributed by horizontal density differences, which leads to baroclinic motions. The other is the adjusting relationship between the internal mode variables \( q, u, v \) and external mode variables \( Q, U, V \), which can be found in CML2012 in detail. On the other hand, the information can also be transferred backwards from internal mode to external mode. The internal mode influences the external mode through the terms \( M_h, M_\phi, R_U \) and \( R_V \) in Eqs. (3b) and (3c). Especially, the first terms on the right side of \( M_h \) and \( M_\phi \) are manifestations on the sea surface contributed by horizontal density differences, which is also known as the influence of the internal tide on the surface tide. A similar interaction exits between the internal and external modes of the adjoint model.

This work is based on the model of CML2012 in which an adjoint internal tidal model was constructed and a much more detailed description about the model discretization can be found.

2.4. Gradients of the cost function with respect to the OBCs

Among all the parameters in this model, the OBCs are the most important and have critical impacts for a regional tidal simulation. Solutions in model interior are uniquely determined by the tidal OBCs once the initial conditions and other parameters have been determined. The work of Lardner et al. [35] discussed the optimal control of OBC in the
channel using a 2-D adjoint tidal model. Zou et al. [75] also developed a sequential open boundary control scheme augmenting radiation conditions and applied it to idealized barotropic wind-driven ocean simulations. In the work of Zhang et al. [71], lateral tidal OBCs that force tides in the internal region are estimated by assimilating predicted coastal tidal elevations into a 2-D POM with the adjoint method. Using the adjoint version of MITGCM, Ayoub [3] carried out experiments to test whether OBC can be constrained by observations inside the domain. Gejade and Copeland [26] and Gejade et al. [27] studied the open boundary control problem for free-surface barotropic Navier–Stokes equations with adjoint data assimilation technology. By assimilating the tidal harmonic constants derived from T/P altimeter data with a 3-D numerical barotropic adjoint tidal model constructed by Zhang and Lu [72], Zhang and Lu [73] optimized the OBCs and simulated the \( M_2 \) tide and tidal current in the Bohai and Yellow Seas.

In this model, the Flather conditions are employed by the external mode and the relaxation conditions are employed by the internal mode (see CML2012). As described in CML2012, the forward model is driven by the barotropic tidal force via the open boundary conditions of the external mode which are given in the form of the Flather conditions as follows:

\[
\eta_1 = \tilde{\eta}' \pm \sqrt{\left(1 - \frac{f^2}{\omega^2}\right) \frac{H}{g} (U - \tilde{U})} \quad \text{and} \quad \eta_1 = \tilde{\eta}' \pm \sqrt{\left(1 - \frac{f^2}{\omega^2}\right) \frac{H}{g} (V - \tilde{V})}
\]  

(9)

where \( \tilde{\eta}' \), \( \tilde{U} \) and \( \tilde{V} \) are external data beyond the model boundary representing, respectively, the clamped surface elevation and horizontal currents relating to the boundary barotropic tidal force, \( H = h + \eta_1 \) is the time-varying total water depth. The sign in Eq. (9) depends on the boundary (positive for eastern and northern boundaries; negative for western and southern boundaries). In this paper, the open boundary conditions here are an adaptation of Eq. (9). Based on the condition (9), we make an assumption that

\[
\tilde{U} = \mp \frac{\tilde{\eta}'}{\sqrt{\left(1 - (f^2/\omega^2)\right) H/g}} \quad \text{and} \quad \tilde{V} = \mp \frac{\tilde{\eta}'}{\sqrt{\left(1 - (f^2/\omega^2)\right) H/g}}
\]  

(10)

whereby \( \tilde{U} \) and \( \tilde{V} \) are absorbed in the clamped surface elevation. In Eq. (10) the sign is positive for western and southern boundaries and is negative for eastern and northern boundaries. In this way, we obtain

\[
\eta_1 = \tilde{\eta} \pm \sqrt{\left(1 - \frac{f^2}{\omega^2}\right) \frac{H}{g} U} \quad \text{and} \quad \eta_1 = \tilde{\eta} \pm \sqrt{\left(1 - \frac{f^2}{\omega^2}\right) \frac{H}{g} V}
\]  

(11)

and it is more simply that only \( \tilde{\eta}'(= 2\tilde{\eta}') \) is independent and needs to be determined in Eq. (11).

Assume that at a certain open boundary grid point \((I, J)\), the \( M_2 \) tidal force at the \( n \)th time step is subject to

\[
\tilde{\eta}_{1,I,J} = a_{I,J} \cos(\omega n \Delta t) + b_{I,J} \sin(\omega n \Delta t),
\]

where \( \omega \) denotes the frequency of \( M_2 \) constituent, \( \Delta t \) is the time step length, \( a_{I,J} \) and \( b_{I,J} \) are the Fourier coefficients as well as the tunable parameters of the model. The gradients of cost function with respect to \( a_{I,J} \) and \( b_{I,J} \) can be deduced from Eq. (8) which yields

\[
\frac{\partial J}{\partial a_{I,J}} + \sum_n T_{I,J}^n \cos(\omega n \Delta t) = 0, \quad \frac{\partial J}{\partial b_{I,J}} + \sum_n T_{I,J}^n \sin(\omega n \Delta t) = 0
\]  

(12)

where

\[
T_{I,J}^n = -\frac{\rho_1}{2R \cos \phi J \Delta \lambda} \left[ q_{I,J,1}^n u_{I,J,1}^n + g \sum_{k=1}^{2} \frac{u_{I,J,k}^n}{\rho_k} (q_{I,J,k}^n + q_{I,J+1,k}^n) \right] + \frac{\rho_1}{2} u_{I,J,1}^n F_{I,J,1}^n
\]

(for grid \((I, J)\) on the western open boundary)

\[
T_{I,J}^n = \frac{\rho_1}{2R \cos \phi J \Delta \lambda} \left[ q_{I,J,1}^n u_{I,J,1}^n + g \sum_{k=1}^{2} \frac{u_{I,J,k}^n}{\rho_k} (q_{I,J,k}^n + q_{I,J+1,k}^n) \right] + \frac{\rho_1}{2} u_{I,J,1}^n F_{I,J,1}^n
\]

(13)
(for grid \((I, J)\) on the eastern open boundary)

\[
T_{I,J}^n = -\frac{\rho_1}{2R\Delta\phi} \left[ \frac{\cos \phi_{J+1/2}}{\cos \phi_{J+1}} q_{d,I+1,J+1}^n v_{I+1,J+1}^n + \sum_{k=1}^{g} \frac{v_{d,I,k}^n}{\rho_k} (q_{l,I,k}^n + q_{l,I+1,k}^n) \right] + \frac{\rho_1}{2} q_{d,I,k}^n F_{v,I,J}^n
\]

(for grid \((I, J)\) on the southern open boundary)

\[
T_{I,J}^n = \frac{\rho_1}{2R\Delta\phi} \left[ \frac{\cos \phi_{J-1/2}}{\cos \phi_{J-1}} q_{a,I,J-1}^n v_{I,J-1}^n + \sum_{k=1}^{g} \frac{v_{a,I,k}^n}{\rho_k} (q_{l,I,J-1,k}^n + q_{l,I,J+1,k}^n) \right] + \frac{\rho_1}{2} v_{a,I,k}^n F_{v,I,J-1}^n
\]

(for grid \((I, J)\) on the northern open boundary)

Here

\[
F_{u,I,J}^n = \frac{u_{I,J+1}^n - u_{I,J-1}^n}{\Delta t} + \frac{u_{I,J+1,1}^n - u_{I,J-1,1}^n}{R \cos \phi_j} \frac{2\Delta\lambda}{2\Delta\phi} + \frac{\bar{v}_{I,J+1,1}^n - \bar{v}_{I,J-1,1}^n}{R} \frac{2\Delta\phi}{2\Delta\phi} - \frac{u_{I,J,1}^n \bar{v}_{I,J,1}^n \tan \phi_j}{R}
\]

\[
F_{v,I,J}^n = \frac{v_{I,J+1}^n - v_{I,J-1}^n}{\Delta t} + \frac{v_{I,J+1,1}^n - v_{I,J-1,1}^n}{R \cos \phi_j} \frac{2\Delta\lambda}{2\Delta\phi} + \frac{\bar{u}_{I,J+1,1}^n - \bar{u}_{I,J-1,1}^n}{R} \frac{2\Delta\phi}{2\Delta\phi} - \frac{v_{I,J,1}^n \bar{u}_{I,J,1}^n \tan \phi_j}{R}
\]

Having determined the adjoint variables \((q_d, u_a\) and \(v_a)\) with the adjoint model, the gradients of cost function with respect to the OBCs (i.e. the Fourier coefficients \(a\) and \(b)\) can thus be calculated. Then the OBCs are optimized with the minimization algorithm described in the following section.

2.5. Optimization with the IPS

Suppose the dimension of the model parameters (the OBCs \(a\) in this work) is \(N\). In realistic oceanic applications, the parameters are supposed to be spatially varying and \(N\) can be large (approximately 2000 or much more). Meanwhile, the observations are often made at a limited number of locations. As a result, the parameter estimation problem might be ill-posed [60], and for identifiability refer to Ref. [45]. The ill-posedness is generally characterized by the non-uniqueness and instability of the parameters in the identification process [67,68]. Chavent [13] studied the uniqueness problem in connection with parameter identification in distributed systems. He noted that the problem is dependent very much upon the type of measurements. A sufficient condition was found for the unique determination of the coefficient in a parabolic system, in which the distributed measurements are available, i.e. measurements are made at every spatial point as a function of time (see [13]). However, distributed measurements are unrealistic and practically do not exist in a field problem. For the case of punctual measurements, i.e. measurements made at a limited number of locations, Chavent [13] demonstrated the intrinsic problem of non-uniqueness.
An alternative is the IPS whereby the dimension of the model parameter space is reduced to a reasonable level. The basic idea of the IPS is simple. Generally, a subset is chosen from the full set of open boundary points as the independent points and the independent OBCs, i.e. OBCs at the independent points, are computed directly with the minimization algorithm. Then the values of OBCs at the full set of open boundary points can be obtained through linear interpolation of these independent ones. The reduction of the dimension of the variable space itself is not a new concept, although it has not been widely used. A dimension reduction technique called the ‘reduced basis approach’ proposed by Parker and Shure [50] has been applied by Egbert and Erofeeva [22] and Kurapov et al. [33] to the representer method, but their purpose was mainly to improve the computation efficiency. Early studies on the spatially varying parameters include the works of Das and Lardner [19,20] which, by carrying out identical twin experiments, concluded that a position dependent drag coefficient and depth correction could be inferred from sea level observations at a number of tidal stations. In order to reduce the dimension of the parameter space, Lardner [34] assumed that the parameters (i.e. the bottom friction coefficient and the water depth in his work) to be estimated are position-dependent and are approximated by piecewise linear interpolations between certain nodal values. Later, Ullman and Wilson [65] studied the spatial variability of the bottom friction coefficient by assimilating the ADCP data into a tidal model of the lower Hudson estuary with the adjoint method. Heemink et al. [29] used an inverse modeling approach to estimate the spatially varying bottom friction coefficient, the spatially varying viscosity and the depth values in a model of the entire European Continental Shelf, in which several parameter groups were selected for a number of sub-domains with the physical properties of the model taken into account. It is worth mentioning the recent works of [40,72,74] in which the bottom friction coefficients at some grid points were selected as the independent parameters, while the bottom friction coefficients at other grid points were obtained through linear interpolation with the independent parameters, and their methodology also included the basic idea of the IPS used in this paper.

Let \( y_j \) be a series of independent OBCs (the Fourier coefficients \( a \) here), \( N_p \leq N \) the number of the independent OBCs and \( y_j \) the results of the linear interpolation of \( x_i \). Then we have the relation:

\[
y_j = \sum_{i=1}^{N_p} \phi_{i,j} x_i
\]

(13)

where \( \phi_{i,j} = W_{i,j}/\sum_{i=1}^{N_p} W_{i,j} \) is the coefficient of linear interpolation and the weighted coefficient \( W_{i,j} \) is calculated in the Cressman form [17], i.e. \( W_{i,j} = (R^2 - r_{ij}^2)/\left(R^2 + r_{ij}^2\right) \), where \( r_{ij} \) is the distance between the grid points of \( x_i \) and \( y_j \), and \( R \) is the influence radius. The gradients of the cost function with respect to the independent OBCs can be derived from \( dL/dx_i = 0 \) which yields \( dJ/dx_i = \sum_{j=1}^{N} \phi_{i,j} (dJ/dy_j) \). Note that the term \( dJ/dy_j \) is computed by Eq. (12).

During assimilation, the cost function is minimized with the L-BFGS method once its gradients with respect to the independent OBCs are known, and meanwhile, the independent OBCs are optimized. CML2012 have demonstrated that the L-BFGS method has a good performance in ideal experiments. In this paper, following CML2012, the L-BFGS version of Liu and Nocedal [39] is employed and a line search subroutine is included to determine the step length. For detailed algorithm for implementing the L-BFGS method refer to the work of [39].

3. Numerical experiments

This work is an extension of the work in CML2012 in which an adjoint internal tidal model was developed and the ideal experiments were employed to test the efficiency and validity of the model on a set of designed Gaussian topographies. In this paper, in order to evaluate the performance of IPS in the OBC estimation, we do continue to follow CML2012 in employing the ideal experiments, but a real topography is further installed here.

In this part, based on the experimental design described in Section 3.1, totally 54 experiments are carried out in four groups and the main results are shown briefly. To highlight the advantage of the IPS, the experiments are arranged as follows. We begin with the first two groups of experiments (Groups 1 and 2) in Section 3.2, in which two sets of observation data are assimilated, respectively, and the IPS is installed in neither group. In Section 3.3, the experiments of Group 3 are carried out and seven versions of IPS are installed. Then from Group 3, six results are picked out as the Group 4 for the six OBC distributions, respectively.
This model is mainly used to simulate the internal tides which play an important role in the mixing and the balance of mechanical energy and heat content in the global ocean [47,55]. As a ubiquitous phenomenon in the ocean, the internal tides are induced in a stratified ocean by the barotropic tidal flow over rough topography such as ridges, trenches, seamounts, and shelf breaks (see [5,31]) and can propagate hundreds of kilometers from their generation regions [18]. Numerous studies (e.g. [47,58]) indicate that the Hawaii Ridge is one of the most important sites where intense internal tidal generation occurs. In the present study, the model is tested in a regional ocean around a section of the Hawaii Ridge, from 198°30' E to 206°50' E and from 17°30' N to 24°10' N (see Fig. 2). The space resolution in this model is 10′×10′ and there are 51×41 horizontal grids totally in this area. The maximum water depth h in this area is 5781 m. The undisturbed interface of this two-layered model is placed at the depth of 1300 m according to the equilibrium position of the observed oscillations of significant amplitude that appeared in previous works (Rudnick et al. [55], their Fig. 2; Levine and Boyd, 2006, their Fig. 4). The angular frequency of $M_2$ tide is $1.4050789025 \times 10^{-4}$ s$^{-1}$ and the whole-time step is 496.863 s (1/90 of the period of $M_2$ tide) for both external and internal modes. The horizontal eddy viscosity coefficient is chosen to be $A_h = 1000$. The coefficients of bottom and interface friction are taken as $\kappa = 0.0025$ and $A_v = 0.003$, respectively.

As shown in Fig. 2, all four boundaries of the computing area are open. It can be implied by numerous studies (e.g. [52]) that the $M_2$ barotropic tide arrives the Hawaii Ridge mainly from the north-east and thus the eastern boundary can be treated as an important resource of the tidal force in the computing area here. In this paper, for simplicity and no loss of generality, we only focus on the inversion of the Fourier coefficient $a$ of the eastern OBCs. Along the eastern boundary, $b(=0)$ is treated as the known and six kinds of prescribed spatial distributions of $a$ are constructed by the trigonometric function

$$a_{I,J} = 0.5 \cdot \sin \frac{2\pi k (J - 3)}{n_y - 1}$$

where $I$ and $J$ are the zonal and meridional indexes of the open boundary points, respectively (in this paper $I = 50$ and $J = 3, 4, \ldots, 39$), $n_y$ is the total number of the open boundary points of interest and here equal to 37, $k$ is the wave number and characterizes different distributions of OBCs. The prescribed values of $k$ for these six distributions are given in Table 1 and the curves of these distributions are presented in Fig. 3. From Fig. 3 we can see that with $k$ increased, the distribution is characterized by more crests and troughs and the OBCs are therefore distributed more complexly in the space domain.
Table 1
Value of $k$ in Eq. (14) and selected optimum IPS (see Section 3.3) for each prescribed distribution of OBCs.

<table>
<thead>
<tr>
<th>OBC distribution</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>0.5</td>
<td>1.0</td>
<td>1.5</td>
<td>2.0</td>
<td>2.5</td>
<td>3.0</td>
</tr>
<tr>
<td>Optimum IPS</td>
<td>C</td>
<td>D</td>
<td>D</td>
<td>C</td>
<td>C</td>
<td>C</td>
</tr>
</tbody>
</table>

The Hawaii Ridge has been concerned as a hot spot of the internal tidal research for some time and abundant observation data is available in this region benefiting from the development of large ocean observing programs such as the Hawaii Ocean Mixing Experiment (HOME) [51] and the remote sensing techniques. In the computing area of the present study, there are many observation locations that have appeared in numerous previous works (e.g. [51,55]) and could be borrowed to our study. However, in the interest of generality, we prefer a random selection of the observation points following CML2012, especially in such idealized experiments. The randomly picked grid points as the observation positions are shown in Fig. 2. The forward model is run with a certain prescribed distribution of OBCs and, assuming the forward model is perfect, the model-generated results of the surface currents (i.e. currents in the upper layer) at these observation points are taken as the pseudo-observations. Then, an initial value (taken as zero here) of $a$ is assigned to run the forward model. Generally in this model, the initial value can be taken as zero and this means that a priori assumption of the OBCs is not required. The difference between the simulated values and pseudo-observations plays the role of the external force of the adjoint model. The optimized OBCs can be obtained through the backward integration of the adjoint equations. The inverse integral time of the adjoint equations is equal to a period of $M_2$ tide. With the procedure above repeated, the OBCs will be optimized continuously and the difference between simulated values and pseudo-observations will be diminished. Meanwhile, the difference between the prescribed and the inverted OBCs will also be decreased.

During the iterative minimization of the cost function, the OBCs are optimized with the L-BFGS method. The iterative minimization terminates once either a convergence criterion is met or the total number of iteration steps reaches

![Fig. 3. Six kinds of prescribed OBCs and the corresponding inverted OBCs for Groups 1, 2 and 4.](image-url)
100. In this paper, the chosen convergence criterion is that the last two values of the cost function are sufficiently close, which is defined by

\[ |J_K - J_{K-1}| < 10^{-8} \quad \text{and} \quad |1 - J_K / J_{K-1}| < 10^{-2}, \]

where \( J_K \) and \( J_{K-1} \) is the last and the second last values of the cost function, respectively.

3.2. Direct inversion without IPS

We begin with a comparison between the first two groups of experiments in which the six kinds of prescribed OBC distributions are inverted by assimilating two sets of observation data, respectively. In Group 1 the surface current data at 30 grid points is taken as the observations and in Group 2 the number of observation points decreases to 10. In both groups, a direct inversion of the OBCs without the IPS is conducted following the methodology described in CML2012.

**Fig. 4** presents the iteration histories of the cost function and its gradient norm (all values are normalized by their own initial values) for the six distributions. The declining cost function indicates that the misfit between the observations and model results is decreased and the observation data has been assimilated efficiently. Meanwhile, the decreased gradient norm suggests that the OBCs are indeed optimized and getting close to the exact solution. By comparing the cost function of the two groups we can see that the assimilation results of Group 1 are better than those of Group 2, which indicates that the assimilation can be more efficient if more observations are used. This is consistent with the conclusion of Chertok and Lardner [16] showing that accurate values of the parameters can be estimated provided that data from a sufficient number of locations are available for assimilation.

The inverted distributions are compared with the prescribed ones in **Fig. 3**. As we can see, all inverted distributions in Group 1 can fit the prescribed ones very well. By contrast, due to the insufficiency of observations, the inversion results in Group 2 are not so satisfactory and some noises are detectable in the numerical solution, which may be considered as an illustration of the problem’s ill-posedness [1]. This means that an accurate inversion of the OBCs is difficult if there is not sufficient observation data available, especially when the number of OBCs is large. It seems that
we have to get a set of satisfactory OBCs at a cost of abundant observations to be made. But through the following experiments we will show that the OBCs can still be estimated very well by assimilating much less observations.

### 3.3. Inversion with IPS

In this section, we will carry out another group of ideal experiments (Group 3), in which the IPS is implemented for each prescribed distribution and the observations at the same 10 points as used in Group 2 are assimilated. In this paper, seven versions of the IPS are considered and they are different in the number of independent points. For simplicity, the independent points are chosen to be distributed at equal intervals along the boundary. The detail of each IPS is listed in Table 2. Actually, in the present study, the influence radius (see Table 2) is equal to the distance between every two adjacent independent points in each scheme. Thus, the experiments employing the Scheme A in Group 3 are indeed equivalent to the ones of Group 2 not employing IPS.

The iteration histories of the cost function and its gradient norm (normalized by their own initial values) for the six prescribed distributions are shown in Figs. 5 and 6, respectively. We can see that both of the cost function and the gradient norm drop dramatically compared with their own initial values, which indicates that the data assimilation and optimization of OBCs are also efficient when the IPS is used. Fig. 5 shows that, for most prescribed distributions, the value of the cost function after assimilation is comparatively large for the IPS in which too many (Scheme A) or too few (Scheme G) independent points are used while the cost function after assimilation has the smallest value for Schemes C or D. This means that 10 or 13 independent points (see Table 2) may be enough to give a satisfactory description of the OBCs that are in the spatial domain less complex than the Distribution 6. Therefore, we take the IPS, with which the experiment has the smallest cost function (see Fig. 5), as the optimum scheme for each prescribed distribution (Table 1), and then gather these six experiments as the Group 4. The inverted distributions of Group 4 are

<table>
<thead>
<tr>
<th>IPS</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N_P)</td>
<td>37</td>
<td>19</td>
<td>13</td>
<td>10</td>
<td>7</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>(R) (in grids)</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>9</td>
<td>18</td>
</tr>
</tbody>
</table>

---

Fig. 5. The iteration history of the cost function for Group 3. The results for the optimum schemes (Table 1) are highlighted with the heavy lines. Note that the value of the cost function has been normalized by its initial value.
presented in Fig. 3 showing that, with the help of the IPS, all the inversion results have been improved a lot compared with the results of Group 2.

Additionally, from Figs. 5 and 6, when focusing on the final results for a certain prescribed distribution, we can find a ubiquitous discrepancy between the magnitude sequences of the cost functions and their gradient norms. For example, the cost function for the Scheme C in the panel of Distribution 1 in Fig. 5 has the smallest value, but the corresponding gradient in Fig. 6 does not yet have the smallest norm. The reason for this discrepancy will be given in Section 4.2.

4. Discussion

4.1. Comparison among Groups 1, 2 and 4

As described in Section 3.2, from Fig. 3 we have seen that when the observation data is sufficient the inversion results are satisfactory for all the prescribed OBCs (see Group 1), but if the available observations are cut down greatly the inversion results may become dissatisfactory due to the strong ill-posedness of the inversion problem (see Group 2). However, Fig. 3 also shows that the inverted distributions of Group 4, in which the IPS is installed, are much better than those of Group 2. For most prescribed distributions, the inversion results of Group 4 are comparable to those of Group 1 although much less observations are assimilated. The inversion error and correlation coefficient between the prescribed and inverted distributions are shown in Fig. 7 for Groups 1, 2 and 4. We can see that without the IPS the results of Group 2 are relatively poor. By contrast, the results of Group 4 employing the IPS become much better and almost at the same level as those of Group 1 in which much more observations are available, and are even better than those of Group 1 for some prescribed distributions. The assimilation errors, i.e., errors between the simulated and the observed surface currents, are compared among the three groups in Fig. 8a. Furthermore, in such idealized experiments in which the ocean states are prescribed and thus known almost everywhere within the computing area, we are allowed to conduct a further comparison between the simulation results and the relevant prescribed ‘true values’ for the surface elevations at data points although not assimilated (see Fig. 8d), as well as for the surface elevations and the currents in the whole upper layer (see Fig. 8b and e) and in the whole lower layer (see Fig. 8c and f). We can see that the result patterns in these three figures are quite consistent with that in Fig. 7 since the simulation results are dominated by the OBCs in this model. To allow a fair comparison, the simulation errors for Group 1 are also computed at the
Fig. 7. (a) Root mean square (RMS) errors and (b) correlation coefficients between the inverted and prescribed OBCs for Groups 1, 2 and 4.

same 10 data points used in Group 2, and this does not change the comparison result above. In conclusion, the IPS is effective and to some extent can remedy the blight of the ill-posedness caused by the insufficiency of observations, which essentially may be due to the reduction of the number of the unknown parameters (OBCs here).

4.2. Comparison among experiments within Group 3

In Section 3.3, simply according to the value of cost function, we have picked out the optimum IPS for each prescribed distribution. The inversion error and correlation coefficient between the prescribed and inverted distributions are shown in Fig. 9. One visible common feature that can be found from both panels of Fig. 9 is that the inversion result is dramatically different along with the different IPS. For a certain prescribed distribution, the inversion result

Fig. 8. Errors between the simulated and model-generated ‘true’ values of tidal (a–c) currents and (d–f) elevations. Panels (a) and (d) are for the observation grid points, (b) and (e) for all wet grid points of the upper layer, (c) and (f) for all wet grid points of the lower layer. In panels (a) and (d) the errors at the 10 observation points are also plotted for the Group 1 (dashed lines) so that the assimilation results of Group 1 can be fairly compared with those of Groups 2 and 4.
is relatively poor when the number of independent points is too large or too small and generally, the best is the result obtained with the Scheme C or D, which is consistent with the case of the cost function values described in Section 3.3. In this section, we will attempt to have a quantitative discussion in detail on the results of Group 3.

On one hand, when too many independent points are installed (such as the Schemes A and B), i.e. the unknown parameters are much more than the observations, according to the theory of ill-posed problems, the non-uniqueness and instability of the solution in the optimization process should result. Some large fluctuations are indeed detectable in the numerical solutions (see Fig. 3), which results in the relatively large discrepancy between the prescribed and inverted OBCs as described in Fig. 9. In addition, from Fig. 9 we can also see that, generally, the inversion error is increased along with the wave number $k$, which can also be seen in Fig. 7a very clearly. This indicates that under a certain given IPS, a higher spatial complexity of the OBCs themselves may increase the degree of difficulty in the OBC inversion, or rather that the spatial complexity of the OBCs can also have a great effect on the model results. To sum up, the inversion result is affected, to a great extent in this model, by the amount of the observation data, the number of the independent unknown parameters (i.e. independent points here) and the degree of the spatial complexity of the prescribed OBCs, whereby we define a function $\mu = (kN_p)/N_O$, where $k$ is the wave number representing the spatial complexity of the prescribed OBCs, $N_p$ the number of the independent points and $N_O$ the number of the observation points. A larger value of $\mu$ suggests a larger inversion error may result, especially when the number of the independent points is large. Then the reciprocal of $\mu$ can be indirectly used to qualitatively and at least partially assess the inversion skill of each experiment of Group 3. The value of $\mu$ is plotted in Fig. 10a for each experiment of Group 3. We can clearly see that, the value of $\mu$ is decreased along with the number of the independent points reduced. When the number of independent points is large, the value of $\mu$ is large and discrepancy between the prescribed and inverted OBCs is also large, which may be considered as a relatively poor inversion skill of the relevant experiments in which too many independent points are used. This may also be used to explain why the inversion error is relatively large when the Scheme A or B is employed as shown in the left half part of Fig. 9a. In addition, $\mu$ is also computed for the experiments of Group 1 and shown in Fig. 10a. We can see that for each prescribed distribution the line of Group 1 intersects the line of Group 3 between C and D, which indicates that when inverting a certain prescribed distribution of OBCs the experiment in Group 3 employing the Scheme C or D should have a result that is close to the result of the relevant experiment in Group 1. This has also been reflected by the comparison between the results of Groups 1 and 4 in both Figs. 7 and 8 and is consistent with the relevant conclusion drawn in Section 4.1.

On the other hand, Fig. 9a also shows that when too few independent points are used (such as the Schemes E, F and G) the inversion error goes up again and the inversion result is neither satisfactory. Obviously, this is due to the fact that too few independent points are not enough to describe the spatial variation of OBCs especially when the OBCs are highly complex in the space domain. Since the full set of OBCs is obtained by interpolating the values at fewer
representative independent points, the fewer independent points are used, the less information can be included in the solution (inverted OBCs). In order to make a quantitative examination, considering the independent OBCs (\(x\) in Eq. (13)) as the variables to be determined, a least squares fitting of Eq. (13) is conducted to each prescribed OBCs for each IPS and the residuals (denoted by \(\epsilon\) here), i.e. the RMS error between the prescribed OBCs and the least squares solution, are presented in Fig. 10b. In general, \(\epsilon\) is increased with the number of the independent points reduced. When the number of independent points is small (refer to Schemes E, F and G), the spatial variation of OBCs cannot be described very well by the sparsely arranged independent points and a relatively large residual \(\epsilon\) is resulted in. But when the number of independent points is large, the fitting result is quite satisfactory and the difference between the least squares solution and the prescribed OBCs is small, especially for the Scheme A in which every open boundary point is treated as an independent point and the distribution of least squares solution completely coincides with the prescribed distribution, which indicates that the scheme using more independent points may have a greater potential for an accurate inversion of the prescribed OBCs. In addition, it should also be noted in Fig. 10b that when using a certain given scheme, in most cases, \(\epsilon\) is also increasingly larger as \(k\) is increased suggesting that a higher spatial complexity of the prescribed OBCs may lead to a larger inversion error, which is also consistent, to a large extent, with results shown in Figs. 9a and 10a.

The RMS error between the inverted OBCs and the corresponding least squares solution (denoted by \(\epsilon_{IL}\)) is also computed for each experiment and compared with the RMS error between the inverted and prescribed OBCs (denoted by \(\epsilon_{IP}\)). And the ratio \(\epsilon_{IL}/\epsilon_{IP}\) is shown in Fig. 11 for each experiment of Group 3. For Scheme A, since the least squares solution is equal to the exact one (the prescribed OBCs), we have the relation \(\epsilon_{IL}=\epsilon_{IP}\) for each prescribed distribution. For the other schemes, we have \(\epsilon_{IL}<\epsilon_{IP}\) and in most cases the ratio \(\epsilon_{IL}/\epsilon_{IP}\) is generally decreased as the independent points are decreased. In fact, in each experiment, the least squares residual \(\epsilon\) (Fig. 10b) is the lower limit of the inversion error \(\epsilon_{IP}\) (Fig. 9a), which demonstrates, in the least squares sense, the inverted OBC distribution should not be better than the least squares solution. Moreover, Fig. 11 suggests that the exact solution (i.e. the prescribed OBCs) cannot be closer to the model solution (i.e. the inverted OBCs) than the least squares solution. Thus, we have found that providing a certain IPS has been determined, the distribution obtained by least squares fitting, rather than the prescribed distribution, is the actual optimal solution to the OBC inversion problem. When the number of the independent points is small, the difference between the actual optimal solution and the exact solution is large, which has in essence determined that the model solution is much closer to the optimal one than to the exact one and the model solution will never be very close to the exact one. This can be considered as an essential cause of the relatively large inversion error shown in the right half part of Fig. 9a, and further more, this is also essentially responsible for the discrepancy between the cost function and its gradient norm described in Section 3.3, i.e. the cost function is large due

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Fig. 10. Values of (a) \(\mu\) and (b) \(\epsilon\) (defined in Section 4.2) for Group 3. In panel (a) the values of \(\mu\) (horizontal straight lines with no plus sign) are also computed for the six experiments of Group 1 and compared with the ones (curves with pluses) for Group 3.
to a large distance between the model and exact solutions while its gradient norm may be small due to a small distance between the model and optimal solutions.

The effect of the spatial complexity of the OBCs denoted by \( k \) on the model results should not be negligible. On one hand, from Figs. 9 and 10 we have known that, for a certain given scheme, a higher spatial complexity of the prescribed OBCs may generally lead to a larger inversion error. On the other hand, the spatial complexity of the OBCs can also dramatically impact the choice of the optimum scheme. From the choice of the optimum scheme for each prescribed distribution (Table 1) we can see that the numbers of the independent points of the optimum schemes for the distributions of lower spatial complexity (Distributions 1, 2 and 3) are not more than those for the distributions of higher spatial complexity (Distributions 4, 5 and 6). In fact, for Distribution 1 specially, the difference between the results obtained with the two Schemes C and D that have the two smallest inversion errors is tiny. The inversion error (0.0047) for Scheme C is indeed only slightly larger than that (0.0038) for Scheme D (Fig. 9a) although Scheme C has gained the advantage according to the value of the cost function (Fig. 5). Therefore, we can conclude that generally more independent points are needed to identify a spatially more complex distribution of OBCs, which is also consistent with our intuition that the more complex a curve is the more representative points are needed to describe this curve and is also supported by further experiments with larger values of \( k \) (not shown here).

In conclusion, as an alternative to reduce the ill-posedness of the inverse problem, the IPS can effectively improve the model result to a certain extent, especially when the observations are sparse. However, as we can see from Fig. 9 and 10, the inversion result, although the IPS has been employed, may still suffer from two unfavorable factors: a large \( \mu \) dominating the inversion error if the number of independent point is too large and a large \( \varepsilon \) dominating the inversion error if the number of independent point is too small, and both may lead to a large inversion error and then a poor model result. Alternatively, we can choose a moderate number of the independent points for which both of the unfavorable factors do not dominate, which lead to a full advantage of the IPS being taken at the cost of an optimal solution approximated to the prescribed OBCs, and hence a compromised solution. Therefore, it is very important to determine an optimal choice of the number of the independent points beforehand or, at least, we should have a reliable criterion according to which the choice can be made. CML2012 suggested that the cost function is reliable and able to reflect the main model results to some extent. This means that the cost function is an good enough criterion so that we can determine the optimum IPS according to the value of the cost function, i.e. the result with the smallest cost function is taken, which is also the way we pick the Group 4 out from the Group 3 in Section 3.3 (see Table 1 and Fig. 5).

4.3. Comparison with regularization method

One classical approach that is often used to solve the ill-posed inverse problems is regularization method. Tikhonov regularization which is pioneered by Tikhonov [63] is one of the most commonly used one. The Tikhonov regularization employs a specific class of so-called “stabilizing functionals” to restrict admissible solutions to spaces of smooth functions [62]. Since this method was proposed, it has been further studied in numerous works (see [23,28,44,64] and
In this section, the IPS suggested in this paper is compared with the Tikhonov regularization when applied to this adjoint assimilation model. First, a smoothing functional is constructed as follows

$$J_{m0} = J + J_{sta}$$

where $J$ is the cost function defined by formula (4), $J_{sta} = (\gamma/2)||\hat{p} - \hat{\hat{p}}||^2$ is the Tikhonov stabilizer, $\gamma$ is a regularization parameter, $\hat{\hat{p}}$ and $\hat{p}$ are prior and optimized model control variables (the OBCs in this paper), respectively.

The Tikhonov regularization is applied to every experiment of Group 2 and the new experiments are named by Group 5. In these experiments, $\hat{\hat{p}}$ is given by the OBCs optimized in the preceding iteration and $\gamma$ is taken as a constant. The minimized cost function and its gradient norm are shown in Fig. 12 and the RMS error and correlation coefficient between the inverted and prescribed OBCs are shown in Fig. 13. The results of Groups 2 and 4 are also presented for comparison in Figs. 12 and 13. As we can see, the regularization method can indeed improve the experiment.

![Fig. 12. (a) Cost function and (b) its gradient norm for Groups 2, 4 and 5.]

![Fig. 13. (a) RMS errors and (b) correlation coefficients between the inverted and prescribed OBCs for Groups 2, 4 and 5.]

references therein).
result to some extent (compared with Group 2). However, its performance is worse than the IPS. The reason may be that, compared with the regularization method, the IPS method has more physical rationality and can yield OBCs maintaining more physical consistency with the tidal model. The classical regularization method has better property than the IPS in mathematical theory and in many practical applications. However, considering more physical natures behind this mathematical problem, the IPS can win greater advantage over the regularization method when applied to this model, although the IPS may fail in some other inverse problems. Therefore, we believe that the IPS could be a useful approach to find a meaningful solution of such inverse problems, at least when applied to simulation of ocean tides.

4.4. Examination of IPS involving Flather condition

In order to prove that the Flather condition with spatially varying OBCs is a sensible control scheme, another two groups of experiments are carried out. To allow a single factor examination of the Flather condition and exclude the influence of other factors especially that of the relaxation boundary condition of the internal mode, a one-layered model (then degenerated into a barotropic system) is used in these experiments. The idea of these experiments is quite simple. Let us consider a squared domain \( \Omega_1 \) (Fig. 14), which can be regarded as an expansion of the domain shown in Fig. 2 (denoted by \( \Omega_2 \)). The open boundaries of this domain can be controlled using a spatially varying \( \eta_1 \) in the Flather condition. Then, a certain flow-field is generated with \( \eta_1 \) (denoted by \( \varphi_1(\Omega_1, t) \)). The values \( \varphi_1 \) at some points inside the domain \( \Omega_2 \) are considered as observations. The point is to recover the OBCs of domain \( \Omega_2 \) (denoted by \( \eta_2 \)) by assimilating these observations. Using \( \eta_2 \), another flow-field within domain \( \Omega_2 \) is generated (denoted by \( \varphi_2(\Omega_2, t) \)). Then, the flow-field \( \varphi_2 \) is compared to the flow-field \( \varphi_1 \) inside domain \( \Omega_2 \).

In this section, two different spatial distributions of \( \eta_1 \) (here the Fourier coefficient \( a \) is assumed to be spatially varying and \( b \) is equal to zero) are considered. Accordingly, the experiments are divided into two groups, G1 and G2 corresponding to the OBCs shown in Fig. 15a and b, respectively. In each group, the seven IPS described in Table 2 and a simplified gradient descent method described in [15] are used to estimate \( \eta_2 \) (both Fourier coefficients \( a \) and \( b \) at all four open boundaries). Then, the relative error \( \varepsilon(\Omega_2, t) = \| \varphi_1 - \varphi_2 \| / \| \varphi_1 \| \) is examined, where \( \| \cdot \| \) is the Euclidean norm. Fig. 16 gives the statistic charts of the relative error calculated with data at all points in domain \( \Omega_2 \) and at all time steps for each experiment. In Fig. 16 each curve is plotted with a function \( f(\varepsilon) \) with respect to \( \varepsilon \), which is defined by

\[
f(\varepsilon) = \frac{N(\varepsilon)}{N(\infty)}
\]

(15)
where $N(\varepsilon)$ represents the number of data whose relative error is smaller than $\varepsilon$. Obviously, $f(\varepsilon)$ means the percentage of data whose relative error is smaller than $\varepsilon$ to the total data number. For every experiment, the percentage of data whose relative error is smaller than 0.1 (i.e., $f(0.1)$) is reported in Table 1. We can see that the relative error in most data is one order of magnitude smaller than the data value itself. This indicates that for both kinds of prescribed $\eta_1$, a majority of the flow-field $\varphi_1$ inside $\Omega_2$ can be recovered quite well by inverting the OBCs $\eta_2$ of domain $\Omega_2$, both with and without IPS (note that the Scheme A is actually equivalent to the scheme that does not employ the IPS).

It is worth noting that the OBCs shown in Fig. 15a drive the tidal flow from the east and therefore the dominant flow direction of is close to the normal to the east boundary of $\Omega_2$ (it cannot be “strictly” normal because the flow direction may be changed to some extent by the topography when it comes to the east boundary of $\Omega_2$). In this case, the suggested control scheme involving the Flather condition works quite well. The reason may be that the Flather condition was derived for a 1D-flow and can support the normal incoming and outgoing waves very well in a 2D-setup. It is well-known that a 3D ocean model should have both normal and tangential velocity components and elevation supplied on the open boundary. There are also some works relating the Flather condition to flows approaching or originating at open boundaries from arbitrary directions (e.g. [10,30]). In practical application, however, the control scheme involving only the normal Flather could be a good approximation, especially when applied to the simulation of ocean tides [42]. In experiments of G2, the specially varying OBCs shown in Fig. 15b are tested, which drive the tidal flow from the east and north whereby we expect to have a flow which is coming into the domain $\Omega_2$ from a direction that is not normal to the boundaries of domain $\Omega_2$. In this case, as we can see from Fig. 16b and Table 3, the flow field can also be recovered quite well, further verifying that the Flather condition with spatially varying Fourier coefficients is a sensible control scheme.

![Fig. 15. Two prescribed distributions of OBCs $\eta_1$ for experiments in (a) G1 and (b) G2.](image)

![Fig. 16. Value of $f$ (defined by formula (15)) with respect to $\varepsilon$ for experiments in (a) G1 and (b) G2. In each panel, one curve corresponds to one IPS.](image)
Table 3
Value of $f(0.1)$ (defined by formula (15)) for experiments in G1 and G2.

<table>
<thead>
<tr>
<th>IPS</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
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<td>G1</td>
<td>64.5</td>
<td>69.6</td>
<td>73.2</td>
<td>76.4</td>
<td>77.4</td>
<td>75.3</td>
<td>53.6</td>
</tr>
<tr>
<td>G2</td>
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<td>84.2</td>
<td>85.2</td>
<td>89.1</td>
<td>80.8</td>
<td>81.2</td>
<td>83.7</td>
</tr>
</tbody>
</table>

5. Summary and conclusions

With a two-layer version of the numerical internal tidal model constructed by CML2012, this paper discusses the application of the adjoint assimilation method to the estimation of the OBCs that are assumed to be spatially varying. Although the OBCs have been inverted successfully in the work of CML2012 as a preliminary study suggesting a great potential of the model, some fluctuations were indeed detectable in the inversion results, which may be considered as an illustration of the problem’s ill-posedness, and more importantly, we find that their results still need to be improved so that the model may have a satisfactory performance in the practical application especially when the observations are sparse. To this end, as an alternative, a dimension reduction technique called the IPS (i.e., a subset is chosen from the full set of open boundary points as the independent points and the OBCs are obtained through linear interpolation of the OBCs at the independent points) is proposed to realize the spatial variation of the OBCs. A series of ideal experiments are carried out on a real topography and with fewer observations (compared with the observations of CML2012) to verify and evaluate the performance and feasibility of the IPS and some results are analyzed quantitatively. The main conclusions from the experimental results are as follows.

On one hand, the IPS is feasible and effective and, to some extent, can remedy the blight caused by the insufficiency of observations. With the independent points being properly arranged the results can be improved dramatically at the level that is comparable to or even better than the results obtained with much more observations to be assimilated. It is important to determine an optimal choice of the number of the independent points beforehand, whereby the advantage of the scheme can be taken as full as possible. In practice, it would be difficult and also unrealistic to find a universally applicable IPS. The results of the ideal experiments in this paper suggest that the choice of the optimum scheme depends, to a large extent, on the spatial complexity of the OBCs, i.e., as the spatial complexity of the OBCs is increased the number of the independent points of the optimum scheme should also increase. And at least, the value of the cost function can be regarded as a good criterion according to which the choice can be made.

On the other hand, the ideal experiments have shown that the IPS can reduce the blight of the ill-posedness of the inverse problem, although not completely, and the inversion results have been improved remarkably. However, this improvement is still at the cost of an optimal solution approximated to the prescribed OBCs, leading to a compromised solution, which means that the exact ‘true values’ of the OBCs may never be found unless every boundary point is arranged as the independent point (this leads to unacceptable results again). Moreover, it should be noted that the use of the IPS greatly relies on an assumption that the values of the OBCs are varying smoothly in the space domain. When the spatial complexity of the OBCs is high, this assumption may fail to hold and the effect of the IPS may be reduced. Fortunately in this internal tidal model, the tidal force contained in the OBCs is imposed from the external mode and the spatial structure of the OBCs is dominated by the barotropic tides which generally have a large wave length in the open ocean. Thus, in a regional ocean domain which will not be too large, the OBCs in our model will not be so complex that the IPS is still to have an excellent performance in the practical application. In this way, the IPS with about ten independent points may be still applicable to the real case in the experiment area of this paper.

In addition, comparison with the Tikhonov regularization indicates that the IPS can win an advantage over the classical regularization method when applied to this model. Therefore, the IPS could be a useful approach to find a meaningful solution of such inverse problems, at least when applied to simulation of ocean tides. Further examination shows that the suggested open boundary control scheme can recover most of the flow-field which is generated from a larger-scale model, indicating that the IPS involving the Flather condition is a sensible control scheme.

In a word, the IPS shows us a way to improve the estimation of the OBCs which play an important role in this model. And we may hope to be also able to improve the efficiency of this scheme. Additionally, particular choices, such as the influence radius and the weighted coefficients of the interpolation, have been made for implementation of
the IPS in this work. However, the performance of the scheme can likely be improved in the application with other options, further highlighting its advantage.

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