Explicit discontinuous spectral element method with entropy generation based artificial viscosity for shocked viscous flows

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ABSTRACT

A spatio-temporal adaptive artificial viscosity (AV) based shock-capturing scheme is proposed for the solution of both inviscid and viscous compressible flows using a high-order parallel Discontinuous Spectral Element Method (DSEM). The artificial viscosity and artificial thermal conduction coefficients are proportional to the viscous and thermal entropy generating terms, respectively, in the viscous entropy conservation law. The magnitude of AV is limited based on the explicit stable CFL criterion, so that the stable artificial viscous time step size is greater than the convective stable time step size. To further ensure the stability of this explicit approach, an adaptive variable order exponential filter is applied, if necessary, in elements where the AV has been limited. In viscous flow computations a modified Jameson's sensor [Ducros et al., 1999 [61]] limits the AV to small values in viscous shear regions, so as to maintain a high-order resolution in smooth regions and an essentially non-oscillatory behavior near sharp gradients/shocks regions. We have performed a systematic and extensive validation of the algorithm with one-dimensional problems (inviscid moving shock and viscous shock-structure interaction), two-dimensional problems (inviscid steady and unsteady shocked flows and viscous shock-boundary layer interaction), and a three-dimensional supersonic turbulent flow over a ramped cavity. These examples demonstrate that the explicit DSEM scheme with adaptive artificial viscosity terms is stable, accurate and efficient.

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1. Introduction

The development of accurate and efficient algorithms for the simulation of high-speed flows in complex geometry has challenged the scientific community for several decades. While much progress has been made, the efficient and accurate Direct Numerical Simulation (DNS) and Large Eddy Simulation (LES) of dynamic environments that include viscous phenomena, turbulence, shocks and the nonlinear interactions in a complex geometry continues to present major challenges. In general, the community agrees, as the first step, that high-order/resolution numerical schemes are necessary to resolve and understand small scale flow phenomena and its interaction with shocks with limited computational resources.

Many high-order/resolution shock-capturing nonlinear schemes have been designed for application on structured meshes with simple geometry. Most noticeably, one of such nonlinear methods is the high-order Weighted Essentially Non-
oscillatory (WENO) scheme [1,2] that combines an essentially non-oscillatory shock capturing ability with high resolution of small scales. For complex geometry, WENO can be applied in combination with the immersed boundary method [3,4] and adaptive mesh refinement [5,6] or through a multi-block mapping approach [7]. For the simulation of multi-dimensional compressible turbulent flows in complex geometry, however, WENO schemes for unstructured mesh (for example [8–11]) are still at their infancy. Moreover, WENO is overly dissipative and computationally inefficient for turbulent flow computation [12–14]. Hybrid numerical schemes that hybridize the nonlinear WENO scheme with low dissipative and more efficient linear (for example, finite difference, spectral, Fourier continuation and compact) schemes [2,15–17] have been developed in recent years, but are yet to be extended to applications with complex geometry.

The literature of high-order shock-capturing methods for unstructured meshes is much less extensive than that of structured mesh. High-order discontinuous Galerkin (DG) or more general Discontinuous Spectral Element Method (DSEM) represent a class of an unstructured mesh method that has received much attention in terms of shock-capturing recently. DSEM combines the advantages of low dispersion and dissipation, boundary fitting for complex geometry and an h/p adaptive mesh approach for high-resolution of small-scale structures. DSEM is cost effective as compared to high-order WENO schemes (see [18] for a brief overview).

In order to suppress Gibb’s oscillations near shocks with higher-order basis functions that DSEM relies on, slope limiters are often employed. The bulk of literature reports on the development and augmentation of explicit Runge Kutta Discontinuous Galerkin (RKDG) methods with min-mod slope limiters introduced by Cockburn [19,20]. The concept of slope limiting is enhanced in [21] by applying a discontinuity detector based on a posteriori error estimates of the solution jump over neighboring element interfaces [22,23]. Works presented in [24–26] report on the usage of both shock detector and slope limiters on unstructured meshes. A dynamic $p$-adaptive gradient based limiting is discussed in [27–29] for unstructured mesh. A positivity-preserving limiter is used in [30] to enforce the minimum entropy principle. Positivity preserving limiting is also proposed in conjunction with 2D mixed element unstructured meshes in [31,32]. In general, limiters are less robust as compared to WENO reconstruction in their control of spurious oscillations in flows with strong shocks. As mentioned in [33], RKDG type implementation experiences difficulty in reducing the residual in steady-state problems. RKDG method combined with WENO limiters are producing promising results [34–40]. However, the performance of these schemes in 3D complex shock-turbulence interactions in complex geometry is yet to be established. The design and application of multi-dimensional limiters for broader application is still an open issue.

A cost-effective alternative to limiting is the artificial viscosity (AV) approach (see [41–44]) where artificial higher, even-order differential terms are added to the equations to dissipate the high frequency waves or smoothen the small scales structures. The major consequence of such approach is that the stable time step required by an explicit time stepping scheme (for example, Runge–Kutta scheme) must be additionally restricted by artificial viscous terms. For shocked flows, the artificial coefficients could become quite large and the resulting time step must be becoming very small just for maintaining the numerical stability.

In the context of DSEM and DG, a number of AV approaches have been developed. A summary of them is presented in Table 1. In the first part of the table, references are listed that have focused on DG and scaled the artificial viscosity in a number of ways based on additional PDE and ODE solutions. The second half of the table lists the contributions pertaining to the entropy viscosity approach. In [45,46] the concept of an AV scaled with the entropy residual (entropy viscosity) was introduced for Continuous/Discontinuous Galerkin method suitable for structured or unstructured mesh. The applicability of the AV in collocation type DSEM methods can be found in [47,48]. Implementation and validation of DSEM has generally been restricted to inviscid flows in two-dimensions. The viscous three-dimensional flow interaction with shocks has not been addressed (as can be deduced from Table 1), which is the main focus of this study.
We develop and assess a high-order parallel DSEM algorithm using an adaptive artificial viscosity approach (DSEM-AV) that captures shocks and provides high-order resolution for both two- and three-dimensional viscous compressible turbulent flows in complex geometries. We propose an estimation of AV scaled with the non-negative entropy generation term in the viscous entropy equation (as opposed to the inviscid entropy residual [45]). The artificial viscosity and thermal conductivity coefficients are designed to be proportional to the entropy generation by viscous and thermal dissipation. To detect high gradients and shocks in each element, a modified Jameson’s (Ducros) shock sensor [61] is employed and the AV coefficients are limited to reduce artificial dissipation in viscous shear regions in order to capture small scale structures such as Kelvin–Helmholtz instability. To remove the severe viscous time step restriction related to a large AV when using an explicit Runge–Kutta scheme, the magnitude of AV is limited to ensure that the stable CFL and viscous time step conditions are of the same order regardless of the AV. To enhance the stability of the numerical scheme in elements containing a strong shock, a spatial-adaptive variable order exponential filter is also used. We have performed a systematic and extensive validation of the DSEM-AV with one-dimensional problems (inviscid moving shock, and viscous shock-structure interaction), two-dimensional problems (inviscid steady and unsteady shocked flows, and viscous shock-boundary layer interaction), and a three-dimensional supersonic turbulent flow over a ramped cavity. These examples demonstrate that the explicit DSEM-AV with adaptive artificial viscosity terms is stable, accurate and efficient.

The paper is organized as follows. After a brief description of the governing equations and 3D DSEM formulation in Sec. 2, the details of the new proposition of artificial viscosity and spectral filter are addressed in Section 3. A wide range of one- and two-dimensional problems is presented as validation to establish the applicability of the formulation, and the results of the three-dimensional supersonic ramped cavity are presented in Sec. 4. Conclusions are given in Sec. 5.

2. Numerical method

In this section, a brief description is given of the governing equations and a flavor of the DSEM that uses a staggered, Chebyshev collocation method to approximate the compressible Navier–Stokes equations. The DSEM method has been extensively discussed and used for the solution of smooth flows [62–66]. We shall present several numerical techniques used in extending the DSEM to non-smooth flows that contain both small scale smooth structures and large scale high gradients/shocks. A spatial-temporal adaptive artificial dissipation is devised by means of a combination of artificial viscosity and thermal conductivity coefficients scaled with viscous entropy generation terms. We will also discuss the limiting of the artificial dissipation in such a way that the stable time step for an explicit Runge–Kutta time stepping scheme can be maintained. An adaptive method that adjusts the dissipation through a shock sensor and a variable order exponential filter is presented that enhances accuracy and stability of the nonlinear PDEs.

2.1. Governing equations

The non-dimensional governing Navier–Stokes conservation laws for a compressible fluid are given by

\[
\frac{\partial \mathbf{Q}}{\partial t} + \nabla \cdot \mathbf{J}^a(\mathbf{Q}) - \frac{1}{Re_f} \nabla \cdot \mathbf{J}^v(\mathbf{Q}) = 0, \tag{1}
\]

and

\[
\mathbf{Q} = \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho E \end{pmatrix}, \quad \mathbf{J}^a(\mathbf{Q}) = \begin{pmatrix} \rho \mathbf{u} \\ \rho \mathbf{uu} + \rho \tilde{\delta} \\ (\rho E + \rho \tilde{\delta}) \mathbf{u} \end{pmatrix}, \quad \mathbf{J}^v(\mathbf{Q}) = \begin{pmatrix} 0 \\ \frac{\tilde{\tau}}{\gamma - 1} \frac{\kappa}{M_f^2 Pr_f} \nabla T \end{pmatrix}, \tag{2}
\]

where \( \mathbf{Q} \) is the conservative variable, \( \mathbf{J}^a \) and \( \mathbf{J}^v \) are the inviscid and viscous fluxes, respectively, \( \rho \) is the density, \( \mathbf{u} \) is the velocity vector, and \( E \) is the total internal energy. \( P \) and \( T \) are the static pressure and temperature, respectively, \( \gamma \) is the ratio of specific heats. \( \tilde{\delta} \) is the Kronecker delta tensor. \( M_f, Re_f, \) and \( Pr_f \) are the reference Mach number, Reynolds number, and Prandtl number, respectively, and \( \kappa \) is the thermal conductivity. The viscous shear stress tensor is given by,

\[
\tilde{\tau} = 2 \mu \tilde{\mathbf{S}} + \left( -\frac{2}{3} \mu \right) (\nabla \cdot \mathbf{u}) \delta,
\]

where \( \tilde{\mathbf{S}} = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \) is the symmetric part of the velocity gradient tensor and the superscript \( ^T \) identifies a transpose. \( \mu \) is the non-dimensional viscosity which is taken unity throughout. The ideal gas equation of state, \( p = \rho T/(\gamma M_f^2) \), closes the equations. In this study, we take \( \gamma = 1.4, \ M_f = 1 \) and \( Pr_f = 0.72 \) for all cases reported here.

2.2. Three-dimensional DSEM: staggered formulation

In the nodal collocation formulation of the DSEM, the approximation begins with the subdivision of the physical domain into \( K \) hexahedral physical elements, \( \Omega_k, k = 1, \ldots, K \). Each physical element is then mapped by an iso-parametric transformation (a linear blending formula) to a unit cubic computational element in the computational domain. Hence, in each computational element, Eqn. (1) becomes,
\[ \frac{\partial \vec{Q}}{\partial t} + \nabla \cdot \vec{F} = 0, \]

where \( \vec{Q} = |\vec{j}| \vec{Q} \), \( \nabla \cdot \vec{F} = \frac{\partial \vec{f}}{\partial \xi} + \frac{\partial \vec{g}}{\partial \eta} + \frac{\partial \vec{h}}{\partial \zeta} \) and

\[
(\vec{f}, \vec{g}, \vec{h})^T = |\vec{j}|^{-1} (\vec{f} - \frac{1}{Re_f} \vec{f}', \vec{g} - \frac{1}{Re_f} \vec{g}', \vec{h} - \frac{1}{Re_f} \vec{h}')^T,
\]

with \(|\vec{j}| = x_\xi (y_\eta z_\zeta - y_\zeta z_\eta) - x_\eta (y_\xi z_\zeta - y_\zeta z_\xi) + x_\zeta (y_\xi y_\eta - y_\eta y_\xi)\) being the determinant of the Jacobian matrix of the domain transformation.

In the staggered grid, spectral formulation, the solution \( \vec{Q} \) is collocated at the Chebyshev–Gauss quadrature points while the fluxes are collocated at the Chebyshev–Lobatto quadrature points. In one dimensional domain \( 0 \leq \xi \leq 1 \), the Gauss \((g)\) points \( \xi_{i+1/2} \) and the Lobatto \((l)\) points \( \xi_i \) of the \( N-1 \) and \( N \) degree Chebyshev polynomial, respectively, are defined by,

\[
\xi_{i+1/2} = \frac{1}{2} \left[ 1 - \cos \left( \frac{i + \frac{1}{2}}{N + 1} \pi \right) \right], \quad i = 0, 1, \ldots, N - 1,
\]

\[
\xi_i = \frac{1}{2} \left[ 1 - \cos \left( \frac{i \pi}{N} \right) \right], \quad i = 0, 1, \ldots, N.
\]

In three dimensions, the Gauss points, denoted by the superscript \( ggg \), is determined by the tensor product of the three one-dimensional Cartesian grids (see Fig. 1). Assuming an equal number of collocation points in each direction, the solution values \( \vec{Q} \) collocated on the \( ggg \) point can be expressed as

\[
\vec{Q}(\xi, \eta, \zeta) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \vec{Q}^{ggg}_{i+1/2, j+1/2, k+1/2} h_{i+1/2}(\xi) h_{j+1/2}(\eta) h_{k+1/2}(\zeta),
\]

where \( h_{p+1/2}(\xi) \) is the Lagrange interpolation polynomial of degree at most \( N-1 \) defined on the Gauss \((g)\) points \( \xi_{m+1/2}, m = 0, \ldots, N - 1 \),

\[
h_{p+1/2}(\xi) = \prod_{m=0, m \neq p}^{N-1} \frac{\xi - \xi_{m+1/2}}{\xi_{p+1/2} - \xi_{m+1/2}}, \quad p = 0, \ldots, N - 1.
\]

Similarly, the fluxes are collocated on the Lobatto \((l)\) points but evaluated at the Gauss \((g)\) points. For example, the \( \xi \)-direction flux, \( \vec{f} \) is approximated on the Lobatto–Gauss–Gauss grid \((ggg)\). The \( \eta \)-direction flux \( \vec{g} \) and \( \zeta \)-direction \( \vec{h} \) are collocated on Gauss–Lobatto–Gauss \((glg)\) grid and Gauss–Gauss–Lobatto \((ggl)\) grid, respectively. Once the fluxes are all computed at the respective Lobatto points and then interpolated at the \( ggg \) points, a classical fourth-order explicit Runge–Kutta time stepping scheme is applied to march the resulting ODE in time with a stable CFL condition (see below).

At the heart of the DSEM algorithm, there are two main operations that interpolate and differentiate a function and transfer the information from one set of quadrature points to the others. These are

1. Interpolate a function \( \vec{F} \) from the Gauss \((g)\) points to Lobatto \((l)\) points using the interpolating polynomial of Eqn. (9) as

\[
\vec{F}^{ggg}_{i, j+1/2, k+1/2}(\xi_i, \eta_{j+1/2}, \zeta_{k+1/2}) = \sum_{p=0}^{N-1} \vec{F}^{ggg}_{p+1/2, j+1/2, k+1/2} h_{p+1/2}(\xi_i).
\]
\[ F^{ggg}(\xi_{l+1/2}, \eta_{j+1/2}, \zeta_{k+1/2}) = \sum_{p=0}^{N-1} F^{ggg}_{l+1/2,p+1/2,k+1/2} h_{p+1/2}(\eta_{j}), \] 
\[ F^{ggg}(\xi_{l+1/2}, \eta_{j+1/2}, \zeta_{k}) = \sum_{p=0}^{N-1} F^{ggg}_{l+1/2,j+1/2,p+1/2} h_{p+1/2}(\zeta_{k}). \] 

2. Differentiate a function \( F \) on Gauss \((g)\) points based on the Lobatto \((l)\) points given in Eqn. (10), namely,

\[ \partial_{\xi} F^{ggg}(\xi_{l+1/2}, \eta_{j+1/2}, \zeta_{k+1/2}) = \sum_{p=0}^{N} F^{ggg}_{p,j+1/2,k+1/2} \partial_{\xi} q_{p}(\xi_{l+1/2}), \]
\[ \partial_{\eta} F^{ggg}(\xi_{l+1/2}, \eta_{j+1/2}, \zeta_{k+1/2}) = \sum_{p=0}^{N} F^{ggg}_{l+1/2,p,k+1/2} \partial_{\eta} q_{p}(\eta_{j+1/2}), \]
\[ \partial_{\zeta} F^{ggg}(\xi_{l+1/2}, \eta_{j+1/2}, \zeta_{k+1/2}) = \sum_{p=0}^{N} F^{ggg}_{l+1/2,j+1/2,p} \partial_{\zeta} q_{p}(\zeta_{k+1/2}), \]

where \( q_{p}(\xi) \) is the Lagrange interpolation polynomial of degree at most \( N \) defined on the Lobatto \((l)\) points along the \( \xi \) direction.

The DSEM algorithm is then summarized as follows:

1. On the Gauss \((g)\) points,
   - Interpolate \( Q \) from the Gauss \((g)\) points to Lobatto \((l)\) points using Eqn. (10).
2. On the Lobatto \((l)\) points,
   - Determine the advective fluxes \( f^{g}, g^{g} \) and \( h^{g} \) using their respective \( Q \) obtained in Step 1.
   - Use a Roe approximate Riemann solver to compute the flux using the two solutions, \( Q \), from the neighboring elements.
   - Apply desired boundary condition for boundary adjacent elements.
   - Compute the derivatives of \( Q, Q', \) on Gauss \((g)\) points using Eqn. (11) and then interpolate them back onto the Lobatto \((l)\) points using Eqn. (10).
     - Determine the viscous flux according to Eqn. (2). To construct a continuous approximation, the average of the viscous fluxes are used on either side of the interface as the interface value.
     - Apply Neumann boundary condition for boundary adjacent elements.
   - Determine the mapped fluxes \( \tilde{f}, \tilde{g} \) and \( \tilde{h} \) using the metric terms of Eqn. (5).
   - Compute spatial derivatives of fluxes at \( ggg \) using Eqn. (11).
3. On the Gauss \((g)\) points,
   - Apply a fourth-order low-storage Runge–Kutta time stepping scheme to update \( Q \) to the next Runge–Kutta stage.
4. Repeat from process above or terminate based on a given termination criteria.

3. Artificial viscosity

Given a hyperbolic conservation law, the general idea of an artificial viscosity method is to add a high even-order dissipation term (with an appropriate sign) to stabilize the numerical scheme in regions where the solution gradients are high relative to the grid spacing, i.e. shock regions. The artificial viscosity must be positive to ensure a dissipative behavior. In a well-posed scheme, the artificial viscosity should vanish for mesh size \( \Delta h \to 0 \) or in regions where the gradient is small relative to the grid spacing. To satisfy these conditions, Guermond et al. [45] scaled the artificial viscosity with the entropy pair inequality or entropy residual, \( (R(D_{h}) = |D_{h}|) \). The entropy residual is theoretically (not necessarily numerically) always greater equal to zero and it tends to zero in regions where the solution is smooth. For the Euler equations or an inviscid ideal gas flow, the entropy is a function of the thermodynamic state variables and satisfies the inequality of the second law of thermodynamics. \( D_{h} = \frac{\partial}{\partial t} s + \nabla \cdot \rho s u \geq 0, \) where \( s \) is the entropy. Irreversible phenomena such as shock-waves lead to entropy production, while in smooth, inviscid and isentropic flows, the entropy residual is zero. The residual, hence, is an excellent candidate for scaling of the artificial viscosity. In [45] the so-called entropy viscosity was used in conjunction with continuous Galerkin method. It was extended for discontinuous Galerkin method [46] and later also exploited for Euler equations and the staggered-grid DSEM [48].

In viscous flow environments or Navier–Stokes simulations, the governing equations are not hyperbolic, but have a mixed hyperbolic–parabolic character. A physical viscous dissipation changes the characteristics of equations and physics. Notably, entropy is generated in areas where viscous stresses are significant such as near walls typically or in general shear regions. In these areas the flow can be smooth, but the entropy residual defined for the inviscid Euler equations is positive because
of viscous entropy generation contributions. With the “inviscid” entropy viscosity definition, the solution is hence excessively dissipated. This is not desirable in turbulence simulations or simulation where subtle mixing of small scales is of importance. In viscous flows, the entropy behavior and its scaling with artificial viscosity, hence, requires a revisit.

To start, in viscous compressible flow, the entropy does not satisfy an inequality, but the entropy conservation equation [67]. The non-dimensional form of this transport equation is derived in Appendix A and is given as

\[
\frac{\partial \rho s}{\partial t} + \nabla \cdot (\rho s \mathbf{u}) - \frac{1}{Pr_f Re_f (\gamma - 1)M_f^2} \nabla \cdot \left( \frac{\alpha}{\gamma^2} \nabla T \right) = 0 ,
\]

where

\[
\Phi = \frac{2}{3} \left( \delta S - \frac{1}{3} \delta S \right) (\delta S - \frac{1}{3} \delta S) ,
\]

\[
\Gamma = \frac{2}{\gamma (\gamma - 1)M_f^2} \left( \frac{1}{Pr_f Re_f (\gamma - 1)M_f^2} \right).
\]

This equation suggests the use of Eqn. (12) as a residual for viscous flow [68] instead of the inviscid entropy residual. However, Eqn. (12) is not an inequality constraint, but a conservation law. In addition to the inviscid terms, the viscous entropy conservation equation contains an additional diffusion transport (second term) and the entropy generation term arising from viscous and thermal dissipations (third term). Combined with the inviscid convective term the second term represents the divergence of an entropy flux vector \( q_i / T \) while the third term measures the irreversibility associated with shock waves and dissipative phenomena. The entropy generation term is \( \Phi + \Gamma \geq 0 \) and in accordance with the second law of thermodynamics known as Clausius–Duhem's inequality. With this the entropy inequality

\[
\frac{\partial \rho s}{\partial t} + \nabla \cdot (\rho s \mathbf{u}) - \frac{1}{Pr_f Re_f (\gamma - 1)M_f^2} \nabla \cdot \left( \frac{\alpha}{\gamma^2} \nabla T \right) \geq 0
\]

follows. Within the continuum assumption, hence, it makes sense to scale the artificial viscosity with the entropy generation terms, which is what we propose and validate in this investigation.

We scale the momentum and thermal conductivity separately with the viscous and conductive entropy generating terms \( \Phi \) and \( \Gamma \) (non-dimensional), respectively, which are both non-negative. The separate scaling is intuitive physically. It improves resolution of physical phenomena where one form of entropy generation is more prominent than the other. Moreover it provides flexibility to change the influence of momentum and energy dissipation if needed. In test below, for example, we find that a lowered thermal conductivity in areas of contact discontinuities yields a sharper capturing of such discontinuities.

Normalizing with the modulus of the entropy norm [45], the non-dimensional artificial viscosity (AV) coefficients are given by

\[
\hat{\mu}_h = C_\mu \frac{\rho (\Delta h)^2}{||\rho s - \overline{\rho s}||_\infty} \left( \frac{\Phi}{T} \right),
\]

\[
\hat{\kappa}_h = C_\kappa \frac{\rho (\Delta h)^2}{||\rho s - \overline{\rho s}||_\infty} \left( \frac{1}{Pr_f (\gamma - 1)M_f^2} \right),
\]

where \( C_\mu \) and \( C_\kappa \) are model parameters, and \( \Delta h \) is the mesh size. \( \overline{\rho s} \) is the spatial average of \( \rho s \) given by,

\[
\rho s = \frac{\rho}{\gamma (\gamma - 1)M_f^2} \ln \left( \frac{p}{p^*} \right).
\]

Following [45], \( ||\rho s - \overline{\rho s}||_\infty \) is the globally (to the computational domain) computed supremum based on the global average entropy.

With this definition, two primary properties of the artificial viscosity are satisfied: Firstly, since the entropy generation term is always positive (theoretically and numerically), the artificial dissipation is positive. Secondly, \( \mu_h \) and \( \kappa_h \) scale with the grid spacing and vanish with \( \Delta h \to 0 \). With this formulation, the artificial viscous and thermal dissipations can be added to the existing viscous flux in the NS equations simply as follows

\[
\frac{1}{Re_f} \left( \begin{array}{c}
0 \\
\frac{1}{3} \frac{\partial}{\partial x} \left[ 4 \left( \dot{\mu}_h Re_f + \mu \right) \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial x} \left[ \frac{\left( \dot{\kappa}_h Re_f + \kappa \right)}{\gamma (\gamma - 1)M_f^2 Re_f} \frac{\partial T}{\partial x} \right]
\end{array} \right),
\]

in the \( x \)-direction, for example.
3.1. Limiting entropy generation in viscous zones

In viscous regions of the flow, such as boundary layers and free shear flows, the entropy generation by viscous and thermal dissipations can be considerable. The artificial viscosity that is scaled with this generation can thus also be large and potentially lead to an overly dissipated solution in those regions. To reduce the artificial dissipation in viscous regions, we employ a modified Jameson’s shock sensor [61,69] (referred here as Ducros sensor)

\[
\theta = \frac{(\nabla \cdot \mathbf{u})^2}{(\nabla \cdot \mathbf{u})^2 + (\nabla \times \mathbf{u})^2 + \epsilon},
\]

(18)
to identify regions with high dilatation, that is, regions with shock waves. The sensor \( \theta \) is within the interval [0, 1], and \( \epsilon = 10^{-12} \) is a small number to avoid division by zero. The value of \( \theta \) is close to unity near shocks and is small for rotation or shear dominated regions. We modify the AV coefficients with the Ducros sensor as follows

\[
\mu_h = \mu_h \theta \mathcal{H}(-\nabla \cdot \mathbf{u}),
\]

\[
\kappa_h = \kappa_h \theta,
\]

(19)
where \( \mathcal{H} \) is the Heaviside function, in order to reduce the contribution of artificial dissipation in rotation dominated regions. The negative dilatation switch essentially removes the contribution of the coefficients in the expansion regions and contact discontinuities, where the divergence of the flow may be significantly negative and the related increase in AV can have an impact on the solution. Expansions regions, however, are always smooth and non-entropy changing (expansion shocks do not exist in the physical world) and hence a contribution from AV is not desired. The Heaviside function \( \mathcal{H}(\cdot) \) ensures that the AV is small in isentropic expansion fans and contact discontinuities. In Sec. 4 below, the effectiveness of the switch is demonstrated with several benchmark problems.

3.2. Explicit time stepping and limiting AV

For explicit time stepping schemes, the viscous terms yield a restriction on the stable time step,

\[
\Delta t_{\text{vis}} = \frac{\Delta h^2}{\nu_{\text{eff}}} \nu_{\text{eff}} R_f,
\]

(20)
where \( \nu_{\text{eff}} = (\mu_h R_f + \mu) / \rho \). This might result in an excessive small time step if the artificial dissipation is large. To alleviate the time step restriction, a computationally expensive implicit time stepping scheme in combination with small grid spacing is often employed when AV methods are used.

Instead, we limit the AV coefficients so that in the inviscid time step

\[
\Delta t_{\text{inv}} = \frac{\nu_{\text{inv}} \Delta h}{(\|\mathbf{u}\| + \sqrt{T})},
\]

(21)
is smaller (more restrictive) than the viscous time step. With \( \nu_{\text{eff}} = \mu h R_f / \rho \) and \( \Delta t_{\text{vis}} \geq \Delta t_{\text{inv}} \), it follows that

\[
\mu_{h, \text{max}} = C_m \rho \Delta h (\|\mathbf{u}\| + \sqrt{T}),
\]

(22)
where \( C_m \propto \nu_{\text{vis}} / \nu_{\text{inv}} \) represents one more model parameter for the upper bound of \( \mu_h \). Similarly, \( \kappa_h \) is bounded by the \( \mu_{h, \text{max}} \).

In this work, we shall define and denote the model constants as \( C_{AV} = (C_\mu, C_\kappa, C_m) \), and the artificial viscosity \( \mu_h \) and thermal conductivity \( \kappa_h \) coefficients as AV coefficients collectively. For briefness, we refer to the combination of the Ducros sensor with switch, simply just as “modified Ducros sensor”.

3.3. Adaptive localized spectral filtering

With the artificial viscosity limited by the convective time step, insufficient dissipation may be available to eliminate high frequency errors from the solution near strong shocks and to prevent instability. The stability of the scheme can be enhanced by applying an adaptive low pass filter to the conservative variables, that is localized around shocks. A low pass filter is an effective technique to remove the high frequency components from the solution in order to stabilize the scheme without affecting the spectral accuracy of the low frequency components of the solution and time step size.

Following Vandeven [70], we define a \( \phi > 1 \) order filter function \( \sigma(k/N) \) as a \( C^\infty([-1, 1]) \) function satisfying

\[
\sigma(0) = 1, \quad \sigma(\pm 1) = 0, \quad \sigma^{(j)}(0) = 0, \quad \sigma^{(j)}(\pm 1) = 0, \quad j \leq \phi,
\]

(23)
where \( \sigma^{(j)} \) denotes the \( j \)-th derivative of \( \sigma \).
The popular $\phi$ order exponential filter is

$$\sigma \left( \frac{k}{N} \right) = \begin{cases} 
\exp \left( -\alpha \left( \frac{k-Nc}{N-Nc} \right)^\phi \right), & \text{if } Nc < |k| \leq N, \\
1, & \text{if } |k| \leq Nc,
\end{cases}$$

(24)

where $\alpha = -\ln \epsilon$ with $\epsilon$ the machine zero, $0 \leq Nc < N$ the cutoff wave number (typically, $Nc = 0$) and $\phi > 1$ the order of the filter. Even though the exponential filter does not strictly adhere to the definition of the filter given above, it does behave asymptotically like one. For the ease of presentation, we discuss the results for the Fourier collocation method, since nothing essential is added in the Chebyshev cases.

Given the Fourier approximation

$$f_N(x) = \sum_{k=-N}^{N} a_k e^{i\pi kx},$$

we construct the following filtered sum by modifying the Fourier coefficients $a_k$ by the filter $\sigma(k/N)$, that is,

$$f_N^\sigma(x) = \sum_{k=-N}^{N} \sigma \left( \frac{k}{N} \right) a_k e^{i\pi kx}.$$  

(25)

The smoothing operation can be cast as a matrix-vector multiplication, where the filtering matrix $S$ is created by substituting the definition of the coefficient $a_k$ into (25) and rearranging the order of the summation, that is

$$f_N^\sigma(x_k) = \sum_{j=0}^{2N-1} S_{kj} f_j,$$

(26)

where $S_{kj}$ are the entries of the filtering matrix $S$, or in the matrix form

$$\tilde{f}_N^\sigma = S \tilde{f}_N.$$  

(27)

An interesting fact about the exponential filter $\sigma$ is that its order can be changed by adjusting the parameter $\phi$. One does not have to provide a different filtering function to obtain a filter of different order. In Fig. 2, the exponential filter with several orders $\phi$ in the wavenumber space is shown. By adapting the exponential filter strength in element with oscillatory solutions stability of the scheme can be improved without unnecessary smoothing in shock-free smooth elements. We have designed a spatially adaptive localized variable-order filtering technique, which adjusts the filter order $\phi$ as a function of entropy viscosity $\mu_h$. If $\mu_h$ is limited by the time step restriction, then a higher filter strength (low order) is employed to reduce oscillations in the solutions. Otherwise, a very mild filter with high order is used to maintain the spectral accuracy of the solution.

3.4. Remarks

We offer the following remarks and observations on the numerical techniques used in the DSEM-AV for inviscid and viscous shocked flow.

- The proposed AV coefficients are cost effective as they can be directly calculated on ggg, lgg, glg or ggl nodes as needed and used to included in the viscous fluxes.
- $\Delta h$ is the local minimum mesh size within an element.
- The $\mu_h$ and $\kappa_h$ on lgg, glg or ggl points are needed for flux computation.
In the present methodology no smoothing operation is necessary needed to apply on $\mu_h$ or $\kappa_h$.

- Eqn. (17) reduces to the inviscid Euler formulation by setting transport properties $\mu = \kappa = 0$.

- The inverse of the non-zero $\mu_h$ is a measure of effective local artificial Reynolds number that is required to regularize a shock, effectively creating a viscous shock, where physical viscosity accounts for the irreversible entropy generation. In the numerical experiments below, we conduct detailed test of the formulation versus an analytical viscous shock solution.

- The AV model constants, $C_{AV}$ should be tuned for optimal stabilization of a given problem. A systematic assessment of the model parameters is presented in Sec. 4.

- $h/p$-adaptivity can potentially be implemented based on the magnitude AV coefficients dictated by the entropy dissipation terms irrespective of smooth/non-smooth zones additional to the shock stabilization.

4. Results and discussion

In the following sub-sections, we validate the proposed artificial viscosity approach, switch and adaptive spectral filter. We split the assessment of the scheme into an inviscid and viscous flow discussion with one- and two-dimensional benchmark problems. A three dimensional supersonic flow over ramped cavity geometry is presented to illustrate the performance of the viscous flow for a three-dimensional, large-scale, viscous, turbulent fluid flow with shocks. A stable CFL=0.9 is taken unless stated otherwise.

4.1. Numerical experiments with inviscid flow

4.1.1. One-dimensional moving shock

We test the effect of model constants $C_{AV}$ on shock capturing in a simple 1D moving shock benchmark. 500 P3 elements are used with the following initial condition:

$$(\rho, u, P) = \begin{cases} (1.4, 0, 1), & -0.5 \leq x < 0, \\ (2.36, 0.57, 2.12), & 0 < x \leq 0.5. \end{cases}$$

We vary $C_{\mu} = C_{\kappa}$ while keeping $C_m = 1$. The pressure $P$, and the AV coefficients $\mu_h$ and $\kappa_h$ are shown in Fig. 3 at $t = 0.2$ when the shock is located at $x_s = 0.28$ according to inviscid exact solution (black dashed curve). Values of $C_{\mu} \approx O(1)$ and $C_m \approx O(1)$ are sufficient to stabilize the moving shock with only minor oscillations around the shock location (tested for polynomial order up to $P_5$ but omitted here). For large $C_{\mu}$, $\mu_h$ is limited by its upper bound according to Eqn. (22). The blue curves (square symbols) identify the maximum allowable dissipation while keeping the explicit stable time step restriction. For $C_{\mu} = 1$, the maximum of $\mu_h \approx 9.82 \times 10^{-4}$, which implies an equivalent artificial local Reynolds numbers $Re_t \gtrsim 1018$ in the shock region. The general trends in the thermal conductivity coefficient $\kappa_h$ are similar to those in $\mu_h$ but with a smaller magnitude.

4.1.2. One-dimensional Sod problem

To demonstrate the performance of artificial dissipation with and without the modified Ducros switch in the presence of shocks, contact discontinuities and expansion waves, we solve the one-dimensional Sod problem using 50 P3 elements with initial condition:

$$(\rho, u, P) = \begin{cases} (0.125, 0, 0.1), & -0.5 \leq x < 0, \\ (1, 0.75, 1), & 0 < x \leq 0.5. \end{cases}$$

We take $C_{AV} = (1, 1, 1)$. Fig. 4 shows the effect of the modified Ducros switch on the density and the AV coefficients. With the modified Ducros switch the prediction of the expansion fan is closer to the exact density solution as compared to the expansion fan predicted without the switch. The switch $\mathcal{H}(-\nabla \cdot \mathbf{u})$ in Eqn. (19) eliminates the contribution of $\mu_h$
in the expansion region, which leads to the improved resolution. The resolution of the contact discontinuity region is in large determined by the entropy generation by the thermal dissipation term. Reducing the artificial thermal conductivity coefficient $\kappa_h$ around the contact discontinuity by lowering $C_\kappa$ yields a sharper contact discontinuity (Fig. 5). It is clear that the independent modification of $\mu_h$ and $\kappa_h$ provides a flexible control that benefit the resolution of different physical behaviors, including shocks, contact discontinuities and expansion waves.

4.1.3. One-dimensional Mach 3 shock interaction with density wave

The performance of the AV formulation is further examined by considering the classical Mach 3 shock-density wave interacting with sinusoidal waves in the density field [71].

$$(\rho, u, P) = \begin{cases} \left( \frac{27}{7}, \frac{4\sqrt{5}}{3}, \frac{31}{2} \right), & 0 \leq x < 1, \\ \left( 1 + 0.2\sin(5x), 0, 1 \right), & 1 \leq x \leq 10, \end{cases}$$

This simplified model mimics the interaction of a shock wave with turbulence.

Fig. 6 shows the density and the AV coefficients of the shock-density wave interaction with 200 $P_3$ elements, $C_{AV} = (1, 0.5, 1)$ at time $t = 1.8$. The DSEM scheme captures the shocklets and resolves high frequency waves behind the main shock and compares well with the reference solution.

4.1.4. Mach 3 flow over a forward facing step

To validate the applicability of the DSEM-AV algorithm on a higher dimensional shocked problem, we simulate the two-dimensional Mach 3 flow over a forward facing step [72]. For details on the setup we refer to extensive descriptions in literature. We use 25,200 $P_3$ elements (each of size 0.01) and a CFL=0.75 with $C_{AV} = (1.0, 0.5, 0.5)$. For time $t \leq 0.1$, we
use a special treatment of the singular point at the expansion corner (which is common practice [72]); we use an eighth order exponential filter in each element and \((\phi = 4)\) is used wherever \(\mu_h\) reaches its local upper bound. Afterwards, the filter order is relaxed to \(\phi = 64\) everywhere except for a few elements in the corner region where we take \(\phi = 4\) regardless of the size of \(\mu_h\). This initial spatial-adaptive filtering prevents instability stemming from the strong reflected bow shock from the corner of the step. Since this reflection is an artificial (unphysical) initialization artifact, the strong adaptive filter is required and justified only initially. At a later time the strong filter is only required around the equally artificial singular corner.

With the modified Ducros switch activated, the temporal evolution of the density field is shown in Fig. 7. The primary flow structures including the shock reflections, expansions are well captured, and perhaps most importantly, the Mach stem is visible in the same locations as reported in literature. As shown in Fig. 8, the AV coefficients trace the bow shock structure and its reflection from the top wall. In the step corner region \(\mu_h\) and \(\kappa_h\) have nominal contributions required for stabilization, but the contributions are local and do not affect the global flow characteristics. The stability of the slip line that emanates from the Mach stem is generally known to be a good measure of the high resolution capability of a numerical scheme. If the dissipation over this slip line is small, then a Kelvin–Helmholtz instability appear. The slip line is stable for an overly dissipative scheme. With the modified Ducros switch deactivated, the slip line is stable because of the entropy generation by shear stresses in the slip line and the associated large, dissipative artificial viscosity. With the modified Ducros switch activated, the artificial dissipation in the shear regions reduces dramatically (see numerical Schlieren in Fig. 9). As the result, the shear layer is resolved with the low dissipative and high resolution DSEM scheme and is unstable as is expected for a high-resolution scheme.
4.2. Numerical experiments with viscous flow

4.2.1. 1D viscous shock-structure

When regularizing an inviscid shock wave with an artificial viscosity approach, the shock dissipation that physically occurs on the scale of the molecular mean free path, is artificially smeared over a finite width on the order of the resolution of the continuum equations. Exactly like a real physical viscous shock, the shock jumps in the far-field correspond to the inviscid Rankine–Hugoniot relations, while the size of the dissipation zone widens with increasing dissipation or viscosity. To understand the relation between the physical dissipation and the artificial dissipation, we compare a regularized inviscid standing shock with an exact Navier–Stokes solution for the one-dimensional viscous shock problem [73]. We initialize the computational domain $x \in [-0.1, 0.1]$ with the exact Navier–Stokes shock structure solution for an upstream Mach 2 flow and $Re_f = 1000$ using 100 $P_3$ elements and a constant $C_{AV} = (1, 1, 1)$. We converge the Euler solution to the steady-state solution, and then compare it to the Navier–Stokes solution.

The inviscid regularized shock matches best with the Navier–Stokes shock structure for $P_4$ elements (see Fig. 10a). With increasing resolution, i.e. a higher $P_n$, the shock is captured more sharply. That is, the shock thickness decreases with increasing order of the polynomial approximation for a given shock jump. With greater $p$-refinement the solution converges to the inviscid solution. The maximum of $\mu_h$ decreases from $P_3$ to $P_6$ elements (for $P_3: \mu_h \approx 1.8 \times 10^{-3}$, $P_4: \mu_h \approx 1.2 \times 10^{-3}$, and $P_6: \mu_h \approx 6.4 \times 10^{-4}$), while the shock thickness increases. $\kappa_h$ behaves similar to $\mu_h$. The $P_3$ solution is smooth, whereas the $P_4$ and $P_6$ solutions show minor oscillations.

The effective local Reynolds number for $P_4$ is approximately $Re_f = 1000$, also corresponding the viscous Navier–Stokes shock solution. This effective Reynolds number is similar to the one reported above in the moving shock discussion. The Reynolds number of 1000 hence appears general. This in turn hints towards a general relation between the regularization, the viscous shock and the constants $C_{AV} = (1, 1, 1)$. We aim to explore this in future work.

4.2.2. 2D Blasius laminar boundary layer

To test the artificial viscosity reducing effect of the modified Ducros switch in a viscous, compressible flow regions, we consider a boundary layer flow over a flat plate under zero pressure gradient with an adiabatic wall boundary condition [74] which has an exact analytical solution. We simulate a Mach 2 flow at a Reynolds number of $Re_f = 2000$. The computational domain size is $[0.4] \times [0.1]$, which is initialized with the analytical compressible Blasius boundary layer profiles, with $100 \times 20 P_3$ elements. Stretched elements near the solid boundary are generated according to $y = L_y f(\eta, \beta)$, where $L_y$ is the computational domain length in the wall normal direction, and the stretching function is given by

$$f(\eta, \beta) = \frac{(\beta + 1) - (\beta - 1)\omega^{1-\eta}}{1 + \omega^{1-\eta}}, \quad \omega = \frac{(\beta + 1)}{(\beta - 1)},$$

where $0 \leq \eta \leq 1$ and $\beta = 1.05$. In this case, $C_{AV} = (1, 1, 0.15)$ and the final time is $t = 72$. 

Fig. 9. Mach 3 flow over a forward facing step: comparison of numerical Schlieren of the temperature field with and without the modified Ducros switch.

Fig. 10. Structure of 1D stationary shock with $p$-refinement. (a) pressure profiles, dotted line indicates inviscid jump and dashed black line is 1D NS exact solution. (b) $\mu_h$ and $\kappa_h$. 

$$f(\eta, \beta) = \frac{(\beta + 1) - (\beta - 1)\omega^{1-\eta}}{1 + \omega^{1-\eta}}, \quad \omega = \frac{(\beta + 1)}{(\beta - 1)},$$

where $0 \leq \eta \leq 1$ and $\beta = 1.05$. In this case, $C_{AV} = (1, 1, 0.15)$ and the final time is $t = 72$. 

$$f(\eta, \beta) = \frac{(\beta + 1) - (\beta - 1)\omega^{1-\eta}}{1 + \omega^{1-\eta}}, \quad \omega = \frac{(\beta + 1)}{(\beta - 1)},$$
The solution \( \rho, u, T \) and AV coefficients \((\mu_b, \kappa_b)\) profiles at the mid-section of the domain are shown in Fig. 11. The viscous stresses inside the boundary layer generate entropy which yields an increased artificial viscosity, \( \mu_b \), and thermal conductivity, \( \kappa_b \), according to Eqns. (15) and (16), respectively. The artificial viscosities are comparable in magnitude to the physical viscosity at \( \approx 5 \times 10^{-4} \) and can clearly not be ignored. With AV, the velocity \( u \) and temperature \( T \) in the boundary layers are diffused more as compared to the analytical solution. The velocity and thermal boundary layers have an increased boundary layer thickness as compared to the analytical one. Because of the increased displacement, the outer flow experiences changes by blockage and boundary condition effects, resulting in a density \( \rho \) increase. The modified Ducros switch is close to zero inside the boundary layer where rotation dominates dilatation. With activated modified Ducros switch, the artificial dissipation inside the boundary layer is thus very small as compared to the physical viscosity, and the analytical profiles of the viscous Navier–Stokes equation are retrieved. Hence, the modified Ducros switch is an important component in the overall DESM-AV algorithm for viscous flows.

4.2.3. 2D shock-wave boundary layer interaction

To rigorously test the DESM-AV scheme for the interaction between shocks and viscous shear flows, we consider a benchmark problem of a shock and a reflected shock interaction with a boundary layer in a shock tube [75–77]:

\[
(\rho, u, v, P) = \begin{cases} 
(1.2, 0, 0, \rho/\gamma), & 0 \leq x < 0.5, 0 \leq y \leq 0.5 \\
(1.2, 0, 0, \rho/\gamma), & 0.5 \leq x \leq 1, 0 \leq y \leq 0.5 
\end{cases}
\]

500 \times 70 P_3 elements are used with stretching parameter \( \beta = 1.05 \) (Eqn. (28)) in wall normal direction. The top boundary is set to symmetric condition while no-slip adiabatic wall boundary conditions are imposed on the remaining boundaries. The Reynolds number is taken \( Re_f = 1000 \), with the AV coefficients \( C_{AV} = (1, 0.1, 0.5) \) and the CFL=0.75. An adaptive spectral filter is necessary with order \( \phi = 16 \) near the regions where \( \max(\mu_b, \kappa_b) \) attains its local upper bound and \( \phi = 64 \) otherwise.

The accelerated flow behind the right moving normal shock leads to the formation of a boundary layer (see Fig. 12). At time \( t = 0.4 \), after reaching the downstream end of the shock tube, the normal shock reflects back and starts interacting with the boundary layer of the bottom wall. This yields a complex shock-wave boundary interaction and associated \( \lambda \) shock, shocklets and vortices at the later time. The reflected shock is strong, which leads to the activation of the stronger adaptive filter, but only in within the immediate vicinity of the shock (Fig. 12a).

A comparison of the numerical Schlieren, and \( \mu_b \) in Fig. 13 illustrates the effect of the modified Ducros switch on the viscous solution. Without the switch the initial boundary layer is thicker (not shown here, but also observed in the laminar boundary layer case above), which affects the shock-boundary layer interaction. With the switch activated the structures are richer and instabilities appear in the slip lines that are not present when the artificial viscosity stabilizes the flow in case of deactivated modified Ducros switch. When the switch is on, the vortices and associated \( \mu_b \) are different in the vortex dominated boundary layer. Detailed shock–shock, shock-vortex and shock-boundary layer interactions are captured and the predicted structure of the foot of the lambda shock, triple point, coiled slip streams with activated modified Ducros switch and are well in accordance with the results in literature [75,76].
4.3. Three dimensional supersonic turbulent flow with a ramped cavity

To demonstrate the applicability of the present methodology in a three-dimensional environment, we simulate the supersonic turbulent flow over a ramped cavity. The geometry of the cavity (see Fig. 14) is taken similar to that presented in [78]. The span-wise dimension and inflow boundary layer thickness are taken same as the step height (unity) of the cav-
ity. The Reynolds number of the incoming flow is based on this reference length-scale. We use the compressible flat plate Mach 2 flow DNS database of [79] for Re = 5000 to initialize the mean three dimensional flow-field and the turbulent fluctuations are reconstructed with second-order statistics according to the stochastic model reported in [80]. The static temperature and pressure are set similar to that reported in [78]. Simulations are carried out with 613,800 P3 elements, having 15 elements in span-wise direction. The AV model constants are \( C_{\text{AV}} = (0.1, 1, 0.15) \) together with the exponential filter order \( \phi = 32 \). To resolve the near wall characteristics of the three-dimensional turbulent supersonic flow, elements are stretched in wall normal (y-axis) direction so that the first Gauss point satisfies \( y^+ \approx 1 \) for this case. The number of Gauss points within the boundary layer thickness \( \delta_{\text{BL}} \) is about 40, and the number of Gauss points within 10% of \( \delta_{\text{BL}} \) is 4.

Fig. 15 shows the general flow features at 18 flow-through times \( (t_{f_f}) \). The positive iso-surfaces of the Q-criterion \( Q \), defined by,

\[
Q = (\Omega_{ij} \Omega_{ij} - \Xi_{ij} \Xi_{ij})/2,
\]

where \( \Omega_{ij} = (u_{ij} - u_{ji})/2 \) and \( \Xi_{ij} = (u_{ij} + u_{ji})/2 \), highlight rotation dominated vortex structures in the flow-field. In this figure, the pressure contours visualize the unsteady oblique shock boundary layer interaction. The downstream stretched and elongated hairpin vortices (large scale structures of the turbulence) are well captured by the present simulation. The incoming three-dimensional turbulent boundary layer evolves as a growing shear layer in the cavity section and impinges with the trailing edge of the cavity, producing an oblique shock wave. The oscillation of the oblique shock wave is closely coupled with the vortex shedding of the shear layer. The distribution of \( \mu_h \) (see Fig. 15) highlight the major shock structures and the controlled artificial dissipation obtained inside the boundary layer. The downstream dissipation appears relatively larger in shock interacted compression regions of the downstream boundary layer. The magnitude of \( \kappa_h \) is similar but with a lower value than that of \( \mu_h \) (not shown).

5. Conclusions

An explicit, shock-capturing, high-order, three-dimensional parallel DSEM-AV flow solver is developed for the simulation of viscous, turbulent flows with shocks in complex geometry. The shock-capturing framework depends on four critical components that include (i) a new artificial viscosity (AV) method that scales the AV with the non-negative entropy generation terms of the viscous entropy transport equation, (ii) a limiting operation using a modified Ducros switch to reduce AV in viscous, shear dominated flow regions, (iii) an AV limiter to ensure that the explicit stable time step restricted by the convective CFL number is obeyed, and (iv) an adaptive localized variable order exponential filter that enhances the stability of the numerical scheme in regions with strong shocks.
The DSEM-AV scheme is systematically tested for inviscid and viscous flow benchmarks in one- and two-dimensions. A three-dimensional supersonic flow over a ramped cavity illustrates the performance of the scheme for a geometric complex domain and for the complex interaction of shocks turbulence and viscous shear regions.

The AV coefficients require some tuning to obtain the optimal results, but generally, we find that the AV model constants $C_\mu$ related to $\mu_h$ can be taken as unity and $C_\kappa$ related to $\kappa_h$ to be between 0.1 and 1, which affects the capturing of shear dominated flows and contact discontinuities. The AV model constant $C_m$ related to the explicitly time step limiting of AV ranges from 0.15 and 1. The usage of variable order exponential filter enhances the stability of the DSEM-AV for problems with strong shocks and singularities.

We find that the activation of the modified Ducros sensor significantly reduces the AV in shear regions and expansion fans. With the reduced AV, the DSEM scheme captures shear instabilities with higher resolution and fine scale structures that are otherwise dissipated.

We show that for a flow with a Reynolds number of 1000, a physical Navier–Stokes viscous shock matches well with an essentially non-oscillatory regularized shock with AV. The inverse of the AV, $1/\mu_h$, which represents an effective artificial Reynolds, is shown to be close to 1000 also. This finding hints that there is a relation between the level of dissipation required, and hence the model constants, an essentially non-oscillatory shock capturing, and/or the general identification of trouble cells. We aim to explore this in future work.

The present AV proposition is not limited to DSEM and is expected to easily extend to other high-order compressible explicit solvers to deal with complex compressible flows involving shocks and turbulence. Detailed study and analysis will be conducted in future with three-dimensional shock turbulence interactions. We will also explore the application of the DSEM-AV based scheme to high speed turbulent flows with chemical reaction.

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Appendix A. Non-dimensionalization of the entropy transport equation

The transport equation of specific entropy $s^*$ in dimensional form and index notation [67] is given by

$$\frac{\partial \rho s^*}{\partial t^*} + \frac{\partial \rho^* s^* u^*_j}{\partial x^*_j} = -\frac{\partial}{\partial x^*_j} \left( \kappa^* \frac{\partial T^*}{\partial x^*_j} \right) - \frac{\Phi^* + \Gamma^*}{T^*} = 0,$$

where,

$$\Phi^* = \tau^*_{ij} \frac{\partial u^*_i}{\partial x^*_j} = 2\mu^* S^*_{ij} S^*_{ij} - \frac{2}{3} \mu^* S^*_{kk} S^*_{ij} + \mu^*_b S^*_{kk} S^*_{ij},$$

$$\Gamma^* = \kappa^* \frac{\partial T^*}{T^*} \frac{\partial T^*}{\partial x^*_j}$$

$$S^*_{ij} = \frac{1}{2} (u^*_i + u^*_j).$$

For $\mu_b = 0$, $\Phi^*$ becomes

$$\Phi^* = 2\mu^*(S^*_{ij} - \frac{1}{3} \delta^*_{ij} S^*_{kk}) (S^*_{ij} - \frac{1}{3} \delta^*_{ij} S^*_{kk}).$$

Following the non-dimensionalization of the Navier–Stokes in Eqs. (1), we have

$$c_f = \sqrt{\gamma RT_f},$$

$$M_f = u_f/c_f = 1,$$

$$Re_f = \frac{L_f u_f \rho_f}{\mu_f},$$

$$Pr_f = \frac{\mu_f C_p}{\kappa_f},$$

$$s_f = u_f^2/T_f.$$
\[
\frac{u_f^2}{C_p f} = M_f^2 (\gamma - 1).
\]  

(A.10)

Now for any property \( \psi^* \), the non-dimensional \( \psi \) is given by \( \psi^* = \psi \psi_f \), where \( \psi_f \) is the reference property. The terms in Eqn. (A.1) therefore are expressed as,

\[
\frac{\partial \rho \psi^*}{\partial t} = \left( \frac{\rho f}{\psi_f} \right) \frac{\partial \rho \psi}{\partial t} = \left( \frac{\rho f u_f}{\psi_f} \right) \frac{\partial \rho \psi}{\partial x_j} \frac{u^*_f}{L_f} \frac{\partial \rho \psi}{\partial x_j}
\]

\[
\frac{\partial}{\partial x_j} \left( \frac{\kappa^*}{T^*} \frac{\partial T^*}{\partial x_j} \right) = \left( \frac{\rho f u_f}{\psi_f} \right) \left( \frac{1}{Re_f Pr_f (\gamma - 1)M_f^2} \right) \frac{\partial}{\partial x_j} \left( \frac{\kappa^*}{T^*} \frac{\partial T^*}{\partial x_j} \right)
\]

\[
\frac{\Phi^*}{T^*} = \left( \frac{\rho f u_f}{\psi_f} \right) \frac{\Phi}{Re_f T} \frac{1}{\gamma^*}
\]

The non-dimensional form in index notation then follows as

\[
\frac{\partial \rho \psi}{\partial t} + \frac{\partial \rho \psi u_j}{\partial x_j} - \frac{1}{Pr_f Re_f (\gamma - 1)M_f^2} \frac{\partial}{\partial x_j} \left( \frac{\kappa^*}{T^*} \frac{\partial T^*}{\partial x_j} \right)
\]

\[
- \frac{1}{Re_f (\Phi^*)} - \frac{1}{Re_f Pr_f (\gamma - 1)M_f^2} \frac{1}{\Gamma^*} \right) = 0,
\]

which becomes Eqn. (12) in vector form.

References


