High order well-balanced finite difference WENO interpolation-based schemes for shallow water equations

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A numerical framework of the generalized form of high order well-balanced finite difference weighted essentially non-oscillatory (WENO) interpolation-based schemes is proposed for the shallow water equations. It demonstrates more flexible construction process than the classical WENO reconstruction-based schemes. The weighted compact nonlinear schemes and finite difference alternative WENO schemes are two specific cases. To maintain the exact C-property, the splitting technique for the source term in the finite difference scheme [Xing and Shu, J. Comput. Phys. 208 (2005)] and the reconstruction technique in the finite volume WENO scheme [Xing and Shu, J. Comput. Phys. 214 (2006)] are adopted. The proposed scheme can be proved mathematically to maintain the exact C-property and demonstrates numerically that it is well-balanced by construction for the stationary water surface. Moreover, the local characteristic projections are employed to further mitigate the Gibbs oscillations. The proposed generic high order WENO schemes not only achieve high order accuracy but also capture the high gradients/shock waves essentially non-oscillatory. Meanwhile, the small perturbation problems can be resolved well on a coarse grid.

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1. Introduction

The finite difference weighted essentially non-oscillatory (WENO) schemes play an important role in solving the hyperbolic conservation laws and other convection dominated partial differential equations. They can produce sharp, non-oscillatory discontinuity transition and reach high order accuracy in the smooth part of the solution. The original WENO scheme was proposed by Liu et al. in their pioneering paper [17], in which a third-order finite volume WENO scheme for solving one-dimension problems was designed. In [7], a generalized framework was provided to design arbitrary high order finite difference WENO schemes, which are more efficient for the multidimensional calculations. Very high order finite difference WENO schemes were well documented in [1].

The high order finite difference WENO schemes [1,7] were designed based on the procedure of ENO [8] and finite volume WENO [17] schemes, which used the flux splitting method to reconstruct the numerical fluxes. We refer to them as the finite difference WENO reconstruction-based scheme. Due to the concise concept, this formulation of numerical fluxes in the high order conservative finite difference schemes has been used widely. However, they have several major drawbacks, such as:

- The schemes can have difficulties implementing the procedure for most two point monotone numerical fluxes because we cannot split each of them to satisfy the upwinding performance for the stability of the schemes [10,16];
- They cannot maintain the free stream solutions exactly in the generalized curvilinear coordinate for multidimensional flow computation. It is because the fluxes involving the metric derivatives cannot achieve the exact metric cancellation [35];
- They are, in general, too dissipative for certain classes of problems (for example, compressible turbulence) [9,12].

In the last two decades, Deng et al. developed the characteristic-wise weighted compact nonlinear (WCN) scheme [5] for solving the hyperbolic conservation laws to enhance the higher resolution and to reduce dissipation of the fine scale structures while capturing shocks essentially non-oscillatory. On one hand, it allows that one can perform the WENO interpolation directly on the point values, i.e. the primitive, conservative or char-
characteristic variables. Hence, the arbitrary monotone numerical flux, such as the Lax-Friedrichs (LF), Roe and HLL/HLLC fluxes, together with the Roe-averaged eigensystem can be employed to compute the interpolation value on the cell-interfaces and its corresponding numerical flux. On the other hand, it can easily provide superior free-stream and vortex preservation on the generalized curvilinear coordinate and the multi-species flow computation [19]. In [4], the authors developed a set of high order hybrid cell-edge and cell-node WCN (HWCN) schemes for the inviscid terms of compressible Navier-Stokes equations. Moreover, a finite difference alternative conservative WENO (AWENO) scheme [10], which is in some sense similar to the WCN scheme, also allows one to use the arbitrary monotone flux to compute the numerical flux on the cell-interfaces and easily provide superior free-stream and vortex preservation on the generalized curvilinear meshes [11]. We refer to them as the finite difference WENO interpolation-based scheme. Readers are referred to [13,20,29,32,33] for the recent development of the WENO interpolation-based schemes.

The shallow water equations play an important role in the modeling and simulation of free surface flows in the rivers and coastal areas, and can predict the tides, storm surge levels, coastline changing from hurricanes [28], when the horizontal length scale is much greater than the vertical length scale. This system admits the stationary solutions, in which the nonzero flux gradients can be exactly balanced by the source terms. However, the nonlinear WENO schemes have difficulties in realizing this purpose due to nonlinearity coming from the nonlinear weights [30] even when the solution is smooth. Therefore, Vukovic and Sopita [27] made an important modification on the classical WENO scheme [7] and applied it to the shallow water equations for preserving the balance between the flux gradients and source terms in a coarse mesh resolution. The schemes [2,26] were verified for maintaining the exact conservation property (C-property), when applied to a quiescent flow with a smooth or non-smooth topography of the sea floor. In [21], Rogers et al. theoretically proved the balancing between the flux gradient and the source term. Xing et al. [30] proposed the high order well-balanced finite difference WENO (reconstruction-based) schemes, which can maintain the exact C-property and achieve genuine high order accuracy for the general solutions of the shallow water equations. Recently, Gao et al. [6] proposed a high order well-balanced WCN scheme for the shallow water equations. In fact, the WCN scheme in [6], which is based on the WENO interpolation for the fluxes, has similar numerical properties to the classical WENO reconstruction-based scheme.

Here, we aim to design a generalized numerical framework of the high order well-balanced WENO interpolation-based schemes for the one- and two-dimensional shallow water equations. The WENO interpolation-based schemes have more flexible construction process than the classical WENO reconstruction-based schemes. The WCN scheme and the AWENO scheme are just two specific cases under this framework. To achieve the high order accuracy, the high order upwind weighted nonlinear interpolation for the conservative variables is employed to compute the fluxes at the cell interfaces. Then high order implicit compact schemes and/or high order derivative terms approximated by explicit central finite difference schemes are employed to obtain the high order approximation of the derivative of fluxes at the cell centers. Different from the well-balanced WENO reconstruction-based scheme, one interpolates the conservative variables to exactly maintain the steady state solution, which can be theoretically proved by employing the splitting technique for the source terms [30] and the reconstruction technique in the finite volume scheme [31]. To further restrain the numerical oscillations around the strong shock waves and high gradients, the local characteristic projection on the conservative variables is also employed. Besides the high order numerical accuracy, well-balanced property and the non-oscillatory behaviors, the good performance on resolving small perturbations is illustrated in several benchmark numerical examples.

The paper is organized as follows. In Section 2, a brief introduction to the generalized formulation of high order WENO interpolation-based schemes is discussed. In Section 3, the high order well-balanced WENO interpolation-based schemes for the shallow water equations are designed and proved to maintain the exact C-property. In Section 4, the exact C-property, optimal convergence order as well as the non-oscillatory property of the proposed scheme are verified by several benchmark numerical examples. Conclusions and future work are given in Section 5.

2. High order WENO interpolation-based schemes

The system of hyperbolic conservation laws (PDEs) without the source terms can be written compactly as

$$\frac{\partial Q}{\partial t} + \nabla \cdot F(Q) = 0.$$  \hspace{1cm} (1)

with the appropriate initial and boundary conditions in the Cartesian domain. We take the one-dimensional case for example and consider a uniformly spaced grid defined by $dx=x_2-x_1<...<x_{N-1/2}<x_{N+1/2}=b$. where $x_{N+1/2}$ are called cell boundaries or interfaces, with cell centers given by $x_i=x_{i-1/2}+\frac{dx}{2}$, where $\Delta x=\frac{b}{N}$ is the uniform grid spacing. The semi-discretized form of Eq. (1) can be transformed into the system of ordinary differential equations (ODEs) and solved by the method of lines

$$\frac{dQ(t)}{dt} = -\frac{\partial F}{\partial x} \Big|_i, \hspace{1cm} i=1,...,N,$$  \hspace{1cm} (2)

where $\frac{\partial F}{\partial x} \Big|_i$ is the numerical approximation to the spatial derivative at the cell centers $x_i$.

2.1. Fifth-order WENO interpolation

A major building block of high order conservative finite difference WENO interpolation-based scheme discussed is the following procedure. Given the point values $f_i=f(x_i)$ (where the $f$ can be primitive, conservative, characteristic variables or the fluxes) of a piecewise smooth function $f(x)$, one needs to find an approximation of $f(x)$ at the cell interfaces $x_{i-1/2}$ as illustrated in Fig. 1. Here, fifth-order ($r=3$) interpolation is taken for example to briefly review the basic idea of high order nonlinear WENO interpolation [22,24].

The five-point $(2r+1=5)$ global stencil $S^5=(x_{i-2},x_{i-1},x_i,x_{i+1},x_{i+2})$ is subdivided into three three-point substencils $S_0=(x_{i-2},x_{i-1},x_i)$, $S_1=(x_{i-1},x_i,x_{i+1})$, $S_2=(x_i,x_{i+1},x_{i+2})$. The fifth-order left-biased polynomial interpolation approximation $f_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}) = \hat{f}(f_{i-2},...,f_{i+2})$ is
built through the convex combination of three third-order interpolation polynomials $f^k(x)$ in each substencil $S_k$, $k = 0, 1, 2$ at the cell interfaces $X_{i±1}$. Therefore, the polynomial interpolation on the global stencil $S^i$ can be written as

$$f_{i±1} = \frac{2}{3} \sum_{k=0}^{2} d_k f^k(x_{i±1}) = \frac{1}{128} \left(3 f_{i-2} - 20 f_{i-1} + 90 f_i + 60 f_{i+1} - 5 f_{i+2}\right), \quad (3)$$

where $(d_0, d_1, d_2) = \left(\frac{1}{16}, \frac{10}{16}, \frac{5}{16}\right)$ are the linear optimal weights and the lower order polynomial interpolation $f^k(x_{i±1})$ are

$$f^0(x_{i±1}) = \frac{1}{8} \left(3 f_{i-1} - 10 f_i + 15 f_{i+1}\right),$$

$$f^1(x_{i±1}) = \frac{1}{8} \left(-f_{i-1} + 6 f_i + 3 f_{i+1}\right),$$

$$f^2(x_{i±1}) = \frac{1}{8} \left(3 f_i + 6 f_{i-1} - f_{i+1}\right). \quad (4)$$

**Remark 1.** This is in contrary to the WENO polynomial reconstruction procedure where the $f(x_{i±1})$ is approximated based on the cell-averaged values of the neighboring cells [3,7,17].

To capture the discontinuities and high gradients without spurious oscillation, the idea of normalized nonlinear weights $\omega_k$ in the WENO interpolation is used to replace the linear weights $d_k$ [34], that is

$$f_{i±1} = \sum_{k=0}^{2} \omega_k f^k(x_{i±1}), \quad (5)$$

where the nonlinear Z-type weights $\omega_k$ are [29]

$$\alpha_k = d_k \left(1 + \left|\frac{T_k}{\beta_k + \epsilon}\right|^p\right), \quad \alpha_k = \frac{\alpha_k}{\sum_{l=0}^{2} \alpha_l}, \quad k = 0, 1, 2. \quad (6)$$

Here, the sensitivity parameter $\epsilon > 0$ is used to prevent the denominator from zero. The power parameter $p \geq 1$ is used to enhance the relative ratio among the local smoothness indicators $\beta_k$, which are defined by

$$\beta_k = \Delta x \int_{X_{i±1}}^{X_{i±1}} \left(\frac{d}{dx} f^k(x)\right)^2 dx + \Delta x^3 \int_{X_{i±1}}^{X_{i±1}} \left(\frac{d^2}{dx^2} f^k(x)\right)^2 dx,$$

$$k = 0, 1, 2. \quad (7)$$

They measure the sizes of the gradient (via the first derivative) and the curative (via the second derivative) of the local lower order polynomial of degree three $f^k(x)$ in each substencil $S_k$. The explicit expressions [7,34] for the smoothness indicators $\beta_k$ are,

$$\beta_0 = \frac{13}{12} (f_i - 2 f_{i+1} + f_{i+2})^2 + \frac{1}{4} (3 f_i - 4 f_{i+1} + f_{i+2})^2,$$

$$\beta_1 = \frac{13}{12} (f_{i-1} - 2 f_i + f_{i+1})^2 + \frac{1}{4} (f_{i-1} - f_{i+1})^2,$$

$$\beta_2 = \frac{13}{12} (f_{i-2} - 2 f_{i-1} + f_i)^2 + \frac{1}{4} (f_{i-2} - 4 f_{i-1} + 3 f_i)^2. \quad (8)$$

These are the same as the classical WENO local smoothness indicators. The parameters $T_5 = |\beta_2 - \beta_0|$, $\epsilon = 10^{-12}$ and $p = 2$ are used in this study.

In summary, the fifth-order WENO interpolation on the left-biased stencil $S^i$ can be expressed as

$$f_{i±1} = \sum_{l=-2}^{2} a_l f_{i+l}, \quad (9)$$

where the coefficients $a_l$ are functions of the WENO nonlinear weights $\omega_k$. In the cases without discontinuity, the coefficients are the same as the weights of the linear upwind central scheme Eq. (3) ($\omega_k = d_k$). We denote the $f_{i±1}^+$ in Eq. (9) by $f_{i±1}^+$ since the global stencil $S^i$ is biased to the left. The procedure to obtain $f_{i±1}^+$ is anti-symmetric to that for $f_{i±1}$ with respect to the target point $X_{i±1}$. We note that the only differences between the polynomial reconstruction- and interpolation-based WENO schemes are the ideal weights $d_k$ and the coefficients in the formula (4). Hence it is straightforward to switch between the two types of WENO schemes without any difficulty.

### 2.2. The generalized form of WENO interpolation-based scheme

A generalized form of WENO interpolation-based scheme is designed in this section. In this formulation, the WENO interpolation of the solution (conservative variables $Q$ in this work) on the cell interfaces $Q_{i±1}$ and the flux $F_{i±1}$ are used to directly compute the numerical flux at the cell-centers $x_i$, which is different from the usual reconstruction procedure of the flux functions. They can be expressed summarily as

$$\Delta x \sum_{l=-K}^{K} \beta_l \frac{\partial F_{i±1}}{\partial x} = \sum_{l=-K}^{K} \beta_l (F_{i±1} - F_{i-l±1}),$$

$$\Delta x \sum_{l=-K}^{K} \gamma_l (F_{i±1} - F_{i-l±1}), \quad (10)$$

where the parameters $K$, $\alpha_k$, $\beta_k$ and $\gamma_l$ are closely related to the accuracy and compact property of the schemes. The equation for approximating the derivative of the flux $\frac{\partial F}{\partial x}$ above consists of three linear and nonlinear terms, which are

(I) the high order global implicit approximation $\frac{\partial F}{\partial x}|_{i±1}$ via the linear compact scheme (local explicit if $\alpha_0 = 0$ for $l \neq 0$);

(II) the high order numerical flux $F_{i±1}$ via the high order nonlinear WENO-interpolation scheme.

(III) the high order numerical approximation of the higher order derivative terms $\frac{\partial^2 F}{\partial x^2}$ in the Taylor expansion of the flux $F_i$ via the lower order linear central finite difference scheme.

**Table 1** shows the parameters in Eq. (10) for several WENO interpolation-based schemes in the literature. The parameter $\alpha$ in the WCN scheme is an adjustable parameter, which can be used to define the order of finite difference compact scheme. For example, the fourth- and sixth-orders compact schemes have $\alpha = \frac{1}{2}$ and $\alpha = \frac{9}{12}$, respectively. The parameter $\beta$ in the HWNC-E scheme is a dissipative parameter, which controls the dissipation of HWNC-E scheme according to the flow features. From Table 1, we can see that the AWENO scheme is a special case of HWNC-E scheme (when $\beta = 1$ in the HWNC-E scheme). Based on the parameters in Table 1, the numerical properties of the generalized WENO interpolation-based scheme Eq. (10) mainly depend on the interpolation of $F_{i±1}$ [18]. In this work, the fifth-order WENO interpolation is used to construct $F_{i±1}$ in term II.

### 2.3. The construction of numerical flux $F_{i±1}$ on the cell interfaces

The numerical flux on the cell interfaces in Eq. (10) is approximated by

$$F_{i±1} = h (Q_{i±1}, Q_{i±1}). \quad (11)$$
with the interpolated values \(Q_{t+\frac{1}{2}}^+\) obtained by the WENO interpolation discussed in the section (2.1). The two-argument function \(h\) is a monotone function and satisfies the following conditions:

- \(h(a, b)\) is a Lipschitz continuous function in both arguments.
- \(h(a, b)\) is a nondecreasing function in \(a\) and a nonincreasing function in \(b\).
- \(h(a, b)\) is consistent with the physical flux \(F\), that is, \(h(a, a) = F(a)\).

More discussions on such monotone fluxes can be found in, e.g., \([10,16]\).

One main advantage of the WENO interpolation-based scheme is that the arbitrary monotone fluxes can be used to compute the fluxes, while the classical WENO reconstruction-based scheme only uses the smooth flux splitting. In this paper, the Lax-Friedrichs (LF) flux and HLL/HLLC flux for the shallow water equations (see Appendix for details) are employed to compute the numerical fluxes on the cell interfaces form the left and right interpolation values \(Q^-\) and \(Q^+\).

### 3. Well-balanced WENO interpolation-based scheme for shallow water equations

The two-dimensional shallow water equations can be written as

\[
\frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial x} + \frac{\partial G(Q)}{\partial y} = S,
\]

with \(Q = (h, hu, hv, h, v)^T\), \(F = (hu, hu^2 + \frac{1}{2}gh^2, hv, hv^2 + \frac{1}{2}gh^2)^T\) and \(G = (hv, hv^2 + \frac{1}{2}gh^2, hu, hu^2 + \frac{1}{2}gh^2)^T\) are vectors of the conservative variables and fluxes. \(h\) is the water depth above the bottom topography. \((u, v)\) is the velocity vector and \(g = 9.812m/s^2\) is the gravitational constant. \(S = (0, -ghv_x, -ghv_y, 0)^T\) is the source term, where \(b(x, y)\) is the vertical height of the bottom topography. In this part, the well-balanced WENO interpolation-based scheme for Eq. (12) will be introduced and theoretically proved to preserve the exact C-property.

The well-balanced WENO interpolation-based schemes with the LF flux for the one-dimensional shallow water equations is introduced so as to maintain the steady state solution \(\phi = h + b = \text{constant}, u = 0\).

By applying the idea in \([30,31]\), the source term \(-ghv_x\) is reformulated equivalently into a sum of two terms \(\frac{1}{2}g(b^2)_x - ghvxb\), where \(b(x)\) is a known smooth/non-smooth function. It should be noted that, at the steady state, the two terms of the split source term \((\frac{1}{2}g(b^2)_x - ghvxb)\) involve only the known function \(b\) and the constants \(g\) and \(\phi\), but not the solutions \(h\) and \(u\). As we will see below, this is crucial for designing the high order well-balanced scheme. The corresponding one-dimensional shallow water equations Eq. (12) become

\[
\frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial x} = S = S_1 + S_2,
\]

where \(Q = (h, hu, hv, h, v)^T\), \(F = (hu, hu^2 + \frac{1}{2}gh^2, hv, hv^2 + \frac{1}{2}gh^2)^T\), and

\[
S_1 = \frac{1}{2}g(0, b^2)_x^., \quad S_2 = -g\phi(0, b)_x^.,
\]

By defining a vector \(B = (b(x), 0)^T\), we define the modified LF numerical flux Eq. (44) as

\[
h^{LF}(Q^-, Q^+) = \frac{1}{2}(F(Q^+) + F(Q^-) - \lambda(Q^{M^-} - Q^{M^+})),
\]

where \(\lambda = \max(Q^+, Q^-)\) (see Appendix for details) and \(Q^0 = Q + B\). The numerical viscosity is now determined by the difference in term of \(Q^M\) instead of the conservative variable \(Q\). This modification is justified since \(b\) is independent of time \(t\). Hence, at the steady state, one has \(Q^{M^-} = Q^{M^+}\) if the same interpolation formula is used for both \(Q\) and \(B\).

We first prove that the WENO interpolation-based scheme with the high order weighted nonlinear interpolation (5) can maintain the exact \(C\)-perty as follows:

**Proposition 1.** The generalized form of WENO interpolation-based scheme (10) with the weighted nonlinear interpolation procedure (5) and the modified LF flux Eq. (16) for the one-dimensional shallow water equations Eq. (14) can maintain the exact \(C\)-property.

**Proof.** For simplicity, we define two vector functions of \(B = (b(x), 0)^T\) as

\[
\zeta(B) = (0, \frac{1}{2}gb^2(x))^T, \quad \eta(B) = (0, -gb(x))^T,
\]

and a vector function of two variables \(Q\) and \(B\) as

\[
G(Q, B) = F(Q) - \zeta(B) - \phi \eta(B),
\]

where

\[
S_1 = \frac{\partial \zeta(B)}{\partial x}, \quad S_2 = \phi \frac{\partial \eta(B)}{\partial x},
\]

we will prove that the generalized scheme Eq. (10) can obtain the exact balance of the flux gradient \(\frac{\partial F(Q)}{\partial x}\) and source terms \(S = S_1 + S_2\) at the steady state Eq. (13), namely

\[
\frac{\partial F(Q)}{\partial x} - S = \frac{\partial G(Q, B)}{\partial x} = 0.
\]

Taking the fifth-order interpolation as an example, \(Q\) on the cell interface \(\chi_{i+\frac{1}{2}}\) are

\[
Q_{i+\frac{1}{2}} = \sum_{l=0}^{2} a_l^+ Q_{i+l}^+ \quad \text{and} \quad Q_{i+\frac{1}{2}}^- = \sum_{l=0}^{2} a_l^- Q_{i+l}^-,
\]

where \(a_l^+\) are the corresponding coefficients, and \(a_l^-\) are anti-symmetric with \(a_l^+\). As noted earlier, they are a function of the WENO nonlinear weights \(\omega_k\). In order to guarantee that the steady state is maintained \(((h + b)_{i+\frac{1}{2}} = \phi)\) and to remove the numerical dissipation in the LF flux \((Q^{M^-} - Q^{M^+}) = 0\), it is necessary and sufficient to require that the same interpolation coefficients \(a_l^+\) and \(a_l^-\) should be applied to find the polynomial interpolation of \(B\) at the cell interface, that is,

\[
B_{i+\frac{1}{2}} = \sum_{l=0}^{2} a_l^+ B_{i+l}^+ \quad \text{and} \quad B_{i+\frac{1}{2}}^- = \sum_{l=0}^{2} a_l^- B_{i+l}^-.
\]
At the steady state Eq. (13), since $\sum_{i=1}^{n-2} a_i^+ + \sum_{i=1}^{n-1} a_i^- = 1$, one has

\[
  h_i^+ + b_i^+ = \sum_{i=1}^{n-2} a_i^+ (h_i + b_i) = \phi, \quad h_i^- + b_i^- = \sum_{i=1}^{n-1} a_i^- (h_i + b_i) = \phi.
\]

Hence, one has

\[
  Q_i^{M+} = (Q + B)_{i+1}^+ = \left( (h + b)_i^+ \cdot (hu)_i^+ \right) = (\phi, 0)^T.
\]

which is a constant at the steady state. The numerical flux $F_{i,1/2}$ on the cell interface $x_{i,1/2}$ in Eq. (16) becomes

\[
  F_{i,1/2} = \frac{1}{2} \left( F(Q_{i+1}^+) + F(Q_{i-1}^-) \right),
\]

since $Q_{i,1/2}^{-} = Q_{i,1/2}^{M+} = 0$.

Similar to the LF flux at the steady state in Eq. (25), we shall also define the functions $\zeta$ and $\eta$ on the cell interface $x_{i,1/2}$ as

\[
  \zeta_{i,1/2} = \frac{1}{2} \left( \zeta(B_{i,1/2}^+) + \zeta(B_{i,1/2}^-) \right),
\]

\[
  \eta_{i,1/2} = \frac{1}{2} \left( \eta(B_{i,1/2}^+) + \eta(B_{i,1/2}^-) \right).
\]

where $\phi_{i,1/2} = \frac{1}{2} (\phi_{i+1}^- + \phi_{i-1}^-) = \phi$ at the steady state. Using Eqs. (25) and (26), through a simple derivation, it can be shown that (see Eq. (18))

\[
  G_{i,1/2} = G_i = \left( 0, \frac{1}{2} g \phi_{i,1/2}^2 \right)^T.
\]
Therefore, it is obvious that
\[ \text{II} = \sum_{i=0}^{K_2} \beta_i \left( G_{i+1, \frac{1}{2}} - G_{i-1, \frac{1}{2}} \right) = 0, \quad \text{III} = \sum_{i=1}^{K_1} \gamma_i \left( G_{i+1} - G_{i-1} \right) = 0. \tag{29} \]

Finally, using the scheme Eq. (10), we have
\[ \frac{\partial G(Q, B)}{\partial x} \bigg|_{i} = 0, \quad i = 1, \ldots, N, \tag{30} \]
in Eq. (20), as required for maintaining C-property. \(\square\)

To maintain the exact C-property in the case of the HLL flux, it should be modified as
\[ \mathfrak{p}_{\text{HLLM}}(Q^+, Q^-) = \begin{cases} F(Q^+), & S^+ \geq 0, \\ \frac{S^- F(Q^+) - S^- F(Q^-) + S^+ (Q^{+z} - Q^{-z})}{S^+ - S^-}, & S^+ < 0 < S^-, \\ F(Q^-), & S^- \leq 0, \end{cases} \tag{31} \]
where \( Q^M = Q + B \). As the Proposition 1, one can obtain the similar proposition as

**Proposition 2.** The generalized form of WENO interpolation-based scheme Eq. (10) with the weighted nonlinear interpolation procedure Eq. (5) and modified HLL flux Eq. (31) for the shallow water equations Eq. (14) can maintain the exact C-property.

One can easily prove Proposition 2 with the modified HLL flux Eq. (31) by using the similar procedure as that used in Proposition 1. Thus, we omit the proof part for saving space.

In the case of the characteristic-wise WENO interpolation-based scheme, we project the conservative variables into the characteristic fields via the left eigenvectors and employ the fifth-order nonlinear WENO interpolation to compute the high order approximation at the cell interfaces, which are then projected back into the physical space via the right eigenvectors. The fluxes (LF or HLL) are used to compute the high order numerical flux on the cell interfaces. Moreover, the same left and right eigenvectors are used to project variables \( B = (b(x), 0)^T \) into the characteristic fields. The same nonlinear interpolation weights, \( \omega_{h} \), computed from the
characteristic variables of \( \mathbf{Q} \) are used for the corresponding variables of \( \mathbf{B} \) after the characteristic projection. Then, one can use the similar procedure as Proposition 1 to prove that the characteristic-wise WENO interpolation-based scheme can maintain the exact C-property. For clarity, we present the algorithm of high order well-balanced characteristic-wise WENO interpolation-based scheme in the following flowchart.

Algorithm 3.1. Given the conservative variables \( \mathbf{Q} \), the flux function \( \mathbf{F}(\mathbf{Q}) \), boundary conditions at time \( t_n \) and the final time \( T \),

- **Step 1:** Split the source terms \( \mathbf{S} \) into two terms \( \mathbf{S}_1 \) and \( \mathbf{S}_2 \) according to the Eq. (14).
- **Step 2:**
  - Project the conservative variables \( \mathbf{Q} \) and \( \mathbf{B} \) into the characteristic fields, the high order nonlinear WENO interpolation is applied to obtain the high order approximation \( \mathbf{Q}_{i+\frac{1}{2}}^p \) and the corresponding nonlinear weights \( \omega_k \) are saved.
  - Use the same nonlinear weights \( \omega_k \) which have been saved to obtain the high order approximation of \( \mathbf{B}_{i+\frac{1}{2}}^p \). Then project \( \mathbf{Q}_{i+\frac{1}{2}}^e \) and \( \mathbf{B}_{i+\frac{1}{2}}^e \) into the physical space via the right eigenvectors.

- **Step 3:** Use the same monotone numerical flux to form the high order approximation of numerical flux \( F_{i+\frac{1}{2}} \), variables \( \zeta_{i+\frac{1}{2}} \) and \( \eta_{i+\frac{1}{2}} \) at the cell-interfaces.
- **Step 4:** Use Eq. (10) to compute the derivatives of flux \( \mathbf{F}(\mathbf{Q}) \), variables \( \zeta(\mathbf{B}) \) and \( \eta(\mathbf{B}) \) respectively.
  - For the WCN scheme, the derivatives at the boundary points are computed by the fifth-order WENO-Z scheme with the well-balanced technique [30] or well-balanced AWENO scheme in this study.
- **Step 5:** Use the third-order TVD Runge-Kutta scheme to update the time integration.
- **Step 6:** If \( t_{n+1} < T \), go to **Step 1**.

Finally, we consider the two-dimensional shallow water equations Eq. (12). The source terms are reformulated as

\[
-ghb_x = \left( \frac{1}{2}gb^2 \right)_x - gbh_x, \quad -ghb_y = \left( \frac{1}{2}gb^2 \right)_y - gbh_y.
\]

The one-dimension procedure described above is followed in the \( x \)- and \( y \)-directions respectively.

**Remark 2.** For the two-dimensional shallow water equations, we use the simple version of the HLLC flux [25] to maintain the exact
Fig. 5. The zoomed figures of water surface level \( h + b \) computed by well-balanced (Left) WCN scheme and (Right) AWENO scheme with and without local characteristic projections at time \( t = 15 \). Schemes (Top) with LF flux and (Bottom) with HLL flux.

Fig. 6. \( L_\infty \) errors in the (Left) water surface level \( h + b \), (Middle) the water discharge \( hu \) and (Right) the water discharge \( hv \) as computed by the WENO interpolation-based scheme at time \( t = 15 \).
C-property in this study. Taking the x-direction for example, the third component of the flux can be expressed in terms of the first component and the variable $v$, that is, $F(x) = F_1 v$. Retaining the HLL flux for the first two components of the flux and combining the velocity $v$ in the y-direction with the first component of the HLL flux. One can obtain an expression for the third component as

$$F(x) = \begin{cases} F_1^{(1)} v^+, & u^* \geq 0, \\ F_1^{(1)} v^-, & u^* < 0, \end{cases}$$

where

$$u^* = \frac{S^+ h^- (u^+ - S^-) - S^- h^+ (u^- - S^+)}{h^+ (u^+ - S^+) - h^- (u^- - S^+)}.$$  

Note that it is only the third component of the flux changes across the middle wave. The HLL flux is adequate for the first and second components of the flux. The advantage is that the exact C-property of the well-balanced schemes based on simple version HLLC flux can be proved straightforward for the shallow water equations once we prove the exact C-property based on the HLL flux. In fact, all results proved in the one-dimensional case, such as the high order accuracy and the exact C-property can be similarly proved in the two-dimensional case.

The resulting system of ordinary differential equations after spatial discretization is advanced in time via the third-order TVD Runge-Kutta scheme [23]. The CFL condition is set to be CFL = 0.45. We will use two specific cases of WENO interpolation-based scheme (e.g. the WCN scheme [5] and the AWENO scheme [10, 29]) to demonstrate their performance and simply refer to the well-balanced WCN scheme and AWENO scheme as the WENO interpolation-based scheme in the following discussion.

4. Numerical results

In this section, we will demonstrate the good performance of the WENO interpolation-based scheme by solving several one- and two-dimensional shallow water equations. The reference solution is computed by the well-balanced WENO scheme [30] by replacing the WENO-JS scheme [7] with the WENO-Z scheme [3]. For clarity, we list the main variables’ unit in Table 2 and omit other variables’ units which can be computed by a simple dimensional analysis.

The $L_2$ and $L_\infty$ errors in the $L_2$ and $L_\infty$ norms, respectively, are defined by

$$L_2 = \sqrt{\Delta x \sum_{i=1}^{N} (f_i^N - f_i^*)^2}, \quad L_\infty = \max_i |f_i^N - f_i^*|,$$

where $f_i^N$ and $f_i^*$ represent the numerical and exactly solutions respectively.

4.1. Orders of accuracy

In this part, the fifth-order optimal convergence order of the WENO interpolation-based schemes is tested for the smooth solution. We apply the following initial conditions

$$b(x) = \sin^2(\pi x), \quad h(x, 0) = 5 + e^{\cos(2\pi x)}, \quad (h u)(x, 0) = \sin(\cos(2\pi x)), \quad x \in [0, 1].$$

and the periodic boundary conditions. The final time is $t = 0.1$. The reference solution is computed by the fifth-order WENO-Z scheme with 6400 cells. $L_2$, $L_\infty$ errors and numerical orders of accuracy for the WENO interpolation-based scheme with different Riemann solvers are given in Tables 3 and 4. The data shows that the proposed scheme achieves the optimal convergence order as those given in [30, 31].

4.2. One-dimensional exact C-property

The exact C-property is the most fundamental and crucial property for a numerical scheme for solving Eq. (14). Thus we follow the classical experiment in [30] to verify that the WENO interpolation-based scheme can maintain the exact C-property. The smooth and discontinuous bottom topographies are chosen as

$$b_1(x) = 5e^{-\frac{x^2}{(x-5)^2}}, \quad b_2(x) = \begin{cases} 4, & \text{if } 4 \leq x \leq 8, \\ 0, & \text{otherwise}. \end{cases}$$

The initial conditions are given by

$$h + b = 10, \quad hu = 0.$$

The computational domain is $x \in [0, 10]$ and the final time is $t = 0.5$. We solve the problem using the scheme from $N = 100$ to $N = 1000$ cells respectively. Theoretically, the stationary solution should be always exactly maintained (C-property).

We use the double precision to perform the computation and show the $L_\infty$ errors for the water surface level $h + b$ and water discharge $hu$ in Fig. 2. It can be clearly seen that the well-balanced schemes behave slightly different, but the $L_\infty$ errors for both the smooth and discontinuous bottom topographies are at the level of round-off errors. In the case of smooth bottom topography, the $L_\infty$ errors computed by non-well-balanced schemes (marked as “SCHEME-NWB” in the figures) converge to the level of round-off error as the grid size increases. However, in the case of discontinuous bottom topography, the $L_\infty$ errors are always around 0(1) due to the existence of discontinuity in the bottom.

4.3. A small perturbation of one-dimensional steady state water

In this section, we consider a quasi-stationary test case given in [14] to demonstrate the capability of the WENO interpolation-based scheme for computation on a rapidly varying flow over a smooth bottom, and the perturbation of a stationary state. The bottom topography consists of a hump

$$b(x) = \begin{cases} 0.25(\cos(10\pi (x - 1.5)) + 1), & \text{if } 1.4 \leq x \leq 1.6, \\ 0, & \text{otherwise}. \end{cases}$$

The initial conditions are

$$h(x, 0) = \begin{cases} 1 - b(x) + \xi, & \text{if } 1.1 \leq x \leq 1.2, \\ 1 - b(x), & \text{otherwise}, \end{cases} \quad u(x, 0) = 0,$$

where $\xi$ is a nonzero constant amplitude of the perturbation. $\xi = 0.2$ and $\xi = 0.001$ are used in this work. The computational domain is $x \in [0, 2]$ and the final time is $t = 0.2$.

According to the classical behavior of wave propagation, the small disturbance will generate two smaller waves propagating to the left and right at the characteristic speeds $\pm \sqrt{gh}$. Many numerical methods have difficulty in the calculations involving such small perturbations of the water surface [14]. We show the water surface level $h + b$ and water discharge $hu$ as computed by the WENO interpolation-based scheme with $N = 200$ cells at time $t = 0.2$ in Fig. 3. The small traveling waves with initial pulse perturbations
are resolved accurately, free of spurious numerical oscillations. Furthermore, the results computed by these schemes agree well with the 3000 cell reference solution and the results given in the literature [14,30].

4.4. One-dimensional dam-breaking problem over a rectangular bump

We choose a classical example in [27], that is the one-dimensional dam-breaking problem over a rectangular bump, to investigate the capability of the WENO interpolation-based scheme in the shock capturing under a more complex condition. It involves an unsteady flow over discontinuous bottom topography

\[ b(x) = \begin{cases} 
8, & \text{if } |x - 750| \leq \frac{1500}{8}, \\
0, & \text{otherwise.}
\end{cases} \]  

(38)

The initial conditions are

\[ h(x, 0) = \begin{cases} 
20 - b(x), & \text{if } x \leq 750, \\
15 - b(x), & \text{otherwise.}
\end{cases} \]

(39)

The computational domain is \( x \in [0, 1500] \) and the final time is \( t = 60 \). We computed the reference solution with 4000 cells in this example.

The initial and final water surface level \( h + b \) and the bottom topography \( b \) are drawn in the left column of Fig. 4. The final water surface level \( h + b \) computed by the proposed schemes with \( N = 500 \) cells at the time \( t = 15 \) and \( t = 60 \) are shown in the right column of Fig. 4 which agrees well with the reference solution and the results given in [15,30]. Although the water depth \( h \) contains discontinuities at \( x = 562.5 \) and \( x = 937.5 \), the numerical solutions are essentially free of oscillations.

The local characteristic projection plays an important role in the shock capturing schemes. For demonstration, the comparative water surface levels of the one-dimension dam-breaking problem computed by the WENO interpolation-based scheme with and without local characteristic projections in local domain \( x \in [620, 950] \) at time \( t = 15 \) are shown in Fig. 5. The numerical schemes without projections are marked as “SCHEME-NP” in the figures. From the zoomed figures, one can easily find that the schemes without the local characteristic projection generate relatively large numerical oscillations around the shock locations.

4.5. Two-dimensional exact C-property

The modified example in [30] is taken to demonstrate that the two-dimensional exact C-property over a hill can be preserved by the WENO interpolation-based scheme. The non-flat elliptical shaped bottom is given by

\[ b(x, y) = 0.8e^{-5(x-0.5)^2-50(y-0.5)^2}. \]  

(40)

The initial conditions are

\[ h(x, y, 0) = 1 - b(x, y), \quad u(x, y, 0) = v(x, y, 0) = 0. \]  

(41)

The computational domain is \([0, 1] \times [0, 1] \) and the final time is \( t = 0.1 \).

The \( L_\infty \) errors in the water surface level \( h + b \) and water discharges \( hu \) and \( hv \) as computed by the WENO interpolation-based scheme at the final time are shown in Fig. 6 for different mesh resolutions from \( N \times M = 100 \times 100 \) to \( N \times M = 1000 \times 1000 \) respectively. In the figures, the lines and symbols in (green and black) blue and red colors are the errors computed by the (non-) well-balanced schemes. One can easily find that the well-balanced schemes can preserve the exact C-property up to the machine rounding error even for a coarse resolution. On the other hand, the \( L_\infty \) errors computed by the non-well-balanced schemes decrease slowly from \( O(10^{-7}) \) to the round-off error with the increasing mesh resolutions.
4.6. A small perturbation of two-dimensional steady-state water

This is a classical two-dimensional example in the literature [14] to demonstrate the capability of the proposed schemes for resolving a perturbation of a stationary state. An isolated elliptical shaped bottom topography is defined by

\[ b(x, y) = 0.8e^{-5(x-0.9)^2-50(y-0.5)^2}. \]  

(42)

The initial conditions are

\[
\begin{align*}
h(x, y, 0) &= \begin{cases}
1 - b(x, y) + 0.01, & \text{if } 0.05 \leq x \leq 0.15, \\
1 - b(x, y), & \text{otherwise},
\end{cases} \\
u(x, y, 0) = v(x, y, 0) &= 0.
\end{align*}
\]  

(43)

Fig. 7. The water surface level \( h + b \) as computed by the well-balanced (Left) WCN scheme and (Right) AWENO scheme with the LF (Red line) and HLLC (Blue line) fluxes respectively at times (From top to bottom) \( t = 0.12, 0.24, 0.36, 0.48 \) and 0.60.
Table 5
The CPU times for the WENO interpolation-based scheme for simulating the small perturbation of two-dimensional steady-state water problem.

<table>
<thead>
<tr>
<th>Scheme</th>
<th>WCN-LF</th>
<th>WCN-HLLC</th>
<th>AWENO-LF</th>
<th>AWENO-HLLC</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPU Time [s]</td>
<td>780</td>
<td>858</td>
<td>690</td>
<td>770</td>
</tr>
</tbody>
</table>

The computational domain is $[0, 2] \times [0, 1]$ and the final time is $t = 0.6$.

We present the water surface level $h + b$ as computed by the WENO interpolation-based scheme with LF and HLLC fluxes under the mesh resolution $600 \times 300$ at times $t = 0.12, 0.24, 0.36, 0.48$ and 0.60 in Fig. 7. The results reach a good agreement with those in the literature [14,15,30]. From those figures, the right-going disturbance propagating past the hump can be clearly observed. The structures of the flow are resolved well by the WENO interpolation-based scheme. Because of the simple flow field structures of the shallow water equation, the results computed by scheme with LF and HLLC fluxes are very similar. However, for the flow field which have more complex and fine structures like the Euler system, the HLLC fluxes will perform better results than the LF fluxes [29].

The corresponding CPU times used by the WENO interpolation-based scheme in this section are listed in Table 5. It is obviously that the AWENO scheme is more efficient than the WCN schemes. It is because in addition to the CPU time required to solve the right hand side of the Eq. (10), the WCN scheme requires to solve a tridiagonal linear equations for the finite difference compact scheme.

5. Concluding and future work

In this work, a generalized form of high order finite difference WENO interpolation-based schemes is proposed and the corresponding well-balanced schemes for the one- and two-dimensional shallow water equations is designed. In that case, the traditional weighted compact nonlinear schemes and finite difference alternative WENO schemes can be regarded as two specific examples of the WENO interpolation-based schemes with a properly chosen set of parameters. By carefully using the splitting technique for the source term and the reconstruction technique in the finite volume WENO scheme, we prove that the well-balanced schemes with the LF and HLL/HLCC fluxes can maintain the exact C-property under the condition of steady state solution. Extensive numerical results show that the well-balanced WENO interpolation-based schemes can reach the optimal convergence order, maintain the exact C-property, resolve the small perturbation problems well and capture the shock waves essentially free of numerical oscillations.

In the future work, we plan to extend the framework of high order generalized well-balanced WENO interpolation-based schemes to solve a class of hyperbolic conservation laws with source terms.

Declaration of Competing Interest

We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work. There is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of, the manuscript entitled. CRediT authorship contribution statement

Peng Li: Data curation, Formal analysis, Writing - original draft.
Wai Sun Don: Data curation, Formal analysis, Writing - original draft.
Zhen Gao: Conceptualization, Formal analysis, Writing - original draft.

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Appendix A

The LF flux and HLL/HLLC flux for the shallow water equations are listed as follows:

- The Lax-Friedrichs (LF) flux is given by

$$h^{LF}(Q^i, Q^j) = \frac{1}{2} \left( F(Q^i) + F(Q^j) - \lambda(Q^i - Q^j) \right).$$

where $\lambda$ is taken as an upper bound over the whole line for $|Q^i|$, in the scalar case, or the absolute value of eigenvalues of the Jacobian matrix in the system case.

- The HLL flux is given by

$$h^{HLL}(Q^i, Q^j) = \begin{cases} F(Q^i), & S^+ \geq 0, \\ \frac{S^+}{Q^+} F(Q^+) + \frac{S^-}{Q^-} F(Q^-) - \sqrt{Q^+ Q^-}, & S^+ < 0 < S^- \\ F(Q^-), & S^- \leq 0. \end{cases}$$

(45)

The wave speed estimates $S^+$ and $S^-$ are computed by

$$S^+ = u^+ - a^+ q^+, \quad S^- = u^- + a^+ q^-,$$

(46)

where $a^+ = \sqrt{gh^+}$, and

$$q^+ = \sqrt{\frac{1}{2}\left(\frac{b^+ - b^-}{b^+ + b^-}\right)}, \quad h^+ > h^+,$$

$$h^+ \leq h^+.$$ (47)

Here $h^+$ is given by

$$h^+ = \frac{1}{2} \left(\frac{1}{2} (a^+ - a^-) + \frac{1}{4} (u^+ - u^-) \right)^2.$$ (48)

Alternative ways to compute the wave speed $S^+$ and $S^-$ can be found in [25].

References


